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A weak Fano 3-fold with del Pezzo fibration of degree 5

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Abstract

This is an explicit construction of the weak Fano 3-fold V with del Pezzo fibration of degree 5 which has the anti-canonical model $\overline{V}$, the double covering of $\mathbf{P}^3$. The 3-fold V has a flop V^+, which is also a weak Fano 3-fold with the same type fibration. The model $\overline{V}$ has 43 double points, the images of flopping curves in V. These 43 flopping curves are devided into three types: one of them is the trisection of the del Pezzo fibration, 8 curves of them are the bisections, and the rest 34 curves are the sections. This work is a part of the classification [5] of smooth weak Fano 3-folds with D-type extremal ray.[1)]

Introduction

This is the remaining part of the report [5] classifying smooth weak Fano 3-folds with del Pezzo fibration. A projective 3-fold V whose anti-canonical divisor $-K_V$ is nef and big is called a *weak Fano 3-fold* ([3]). Here a divisor D on V is *nef* if $(D, C) \geq 0$ for any effective curve C in V, and D is *big* if it has a positive self-intersection-number $(D^3) > 0$. Let V be a projective 3-fold, and C a curve. A surjective morphism $\pi : V \to C$ is a *del Pezzo fibration of degree d* if its general fiber is a del Pezzo surface of degree d.

We construct a smooth weak Fano 3-fold V with del Pezzo fibration of degree 5 whose anti-canonical model $\overline{V}$ is of degree $(-K_{\overline{V}})^3 = 2$. We have already seen in [5] that there is a series of smooth weak Fano 3-folds with del Pezzo fibration of degree 5, and that the series is composed of 11 weak Fano 3-folds whose anti-canonical divisor has the self-intersection-number $2g$, $g = 1, 2, \cdots, 11$. These models constructed as the subbundle $V \subset X \to \mathbf{P}^1$ in [5]. Here we study one of them more precisely, using an explicit representation by the bihomogeneous coordinates system of X.

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The weak Fano 3-fold V

In this section, we give the explicit representation of our weak Fano 3-fold V by using the bihomogeneous coordinates system of the $\mathbf{P}^5$-bundle containing V.

Let $\mathcal{E}$ be a locally free sheaf ${\mathcal{O}_{\mathbf{P}^1}}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus 4}$ of rank 6 over $\mathbf{P}^1$, and let $\pi : X = \mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$ be the $\mathbf{P}^5$-bundle over $\mathbf{P}^1$ associated to $\mathcal{E}$. Let $R = \mathbf{C}[x_0, x_1, y_2, y_3, y_4, y_5, t_0, t_1]$ be the bihomogeneous coordinates ring of X with bidegrees $\deg x_i = (1,0)$, $\deg y_i = (1,-1)$, $\deg t_j = (0,1)$. Consider the subvariety $V \subset X$ defined by the Pfaffian Pfaff(M) of the skew-symmetric matrix

$$M = \begin{pmatrix} 0 & g & x_0 & x_1+a & b \\ -g & 0 & x_1 & c & x_0+d \\ -x_0 & -x_1 & 0 & y_2 & y_3 \\ -x_1-a & -c & -y_2 & 0 & y_4 \\ -b & -x_0-d & -y_3 & -y_4 & 0 \end{pmatrix}.$$

Here

$$\begin{cases} g = g_0{t_0}^2 + g_1 t_0 t_1 + g_2{t_1}^2 \\ a = a_0 t_0 + a_1 t_1 \\ b = b_0 t_0 + b_1 t_1 \\ c = c_0 t_0 + c_1 t_1 \\ d = d_0 t_0 + d_1 t_1 \end{cases}$$

are the bihomogeneous polynomials in y_i and t_j with $\deg g = (1,2)$, $\deg a = \deg b = \deg c = \deg d = (1,1)$. If these polynominals are general, the variety V is smooth. Every fiber of $\pi : V \to \mathbf{P}^1$ is defined by the Pfaffian Pfaff$(M|_{t=\tau})$ for $\tau \in \mathbf{P}^1$, whence $\pi : V \to \mathbf{P}^1$ is a del Pezzo fibration of degree 5. The subvariety $V \subset X$ is defined by the system of equations

$$\begin{cases} f_5 = gy_2 - x_0 c + x_1(x_1 + a) \\ f_4 = gy_3 - x_0(x_0 + d) + x_1 b \\ f_3 = gy_4 - (x_1 + a)(x_0 + d) + bc \\ f_2 = x_0 y_4 - (x_1 + a)y_3 + by_2 \\ f_1 = x_1 y_4 - cy_3 + (x_0 + d)y_2 \end{cases} \qquad \cdots\cdots\cdots \qquad \text{(S)}$$

in the bihomogeneous corrdinates ring R, i.e., $V = \mathbf{Proj}\ R/(f_1, \cdots, f_5)$.

In order to see that V is a weak Fano 3-fold, we compute the anti-canonical divisor $-K_V$ and its intersection numbers, and deduce the nef-and-bigness for $-K_V$.

The Picard group Pic$X$ is generated by the tautological bundle H and a fiber F of $\pi : X \to \mathbf{P}^1$, and the canonical divisor of X is $K_X \sim -6H + 2F$. Let

$V_1 = (f_1, f_2, f_3)$, $V_2 = (f_1, f_2, f_3, y_4)$, $V_3 = (f_1, f_2, y_4)$, and $V_4 = (y_2, y_3, y_4)$ be subvarieties in X of dimension 3. Then we have $V_1 = V \cup V_2$ and $V_3 = V_2 \cup V_4$, i.e., $V \sim V_1 - V_2 \sim V_1 - (V_3 - V_4)$ in the Chow ring Chow(X) of X. The varieties V_1, V_3 and V_4 are complete intersections, and

$$V_1 \sim (2H - F) \cdot (2H - F) \cdot 2H \sim 8H^3 - 8H^2F,$$
$$V_3 \sim (2H - F) \cdot (2H - F) \cdot (H - F) \sim 4H^3 - 8H^2F,$$
$$V_4 \sim (H - F) \cdot (H - F) \cdot (H - F) \sim H^3 - 3H^2F$$

in Chow(X), hence we have

$$V \sim 5H^3 - 3H^2F.$$

Since the canonical divisor of V_1 is

$$K_{V_1} \sim (K_X + (2H - F) + (2H - F) + 2H)\big|_{V_1} \sim 0$$

and the conductor C_V on V is $V_2\big|_V \sim H_V - F_V$, it follows that

$$-K_V \sim -\left(K_{V_1}\big|_V - C_V\right) \sim H_V - F_V.$$

Therefore, the self-intersection-number is

$$\begin{aligned}(-K_V)^3_V &= (H_V - F_V)^3_V \\ &= ((H - F)^3 \cdot (5H^3 - 3H^2F))_X = 5(H^6)_X - 18(H^5F)_X \\ &= 20 - 18 = 2 > 0.\end{aligned}$$

Hence we have only to see the nefness of $-K_V$.

Let s_0 be the minimal section of $\pi : X \to \mathbf{P}^1$, i.e., the section corresponding to the exact sequence $\mathcal{E} \to \mathcal{O}_{\mathbf{P}^1} \to 0$, and let l be a line in any fiber of $\pi : X \to \mathbf{P}^1$. Let $C \subset V$ be an irreducible curve with $(C \cdot -K_V)_V \leq 0$. Every effective curve C in V is considered the effective curve $C \equiv al + bs_0$ in X for some non-negative integers a and b. Then the intersection numbers with divisors H_V and F_V are $(C \cdot H_V)_V = (C \cdot H)_X = a$ and $(C \cdot F_V)_V = (C \cdot F)_X = b$, hence $(C \cdot -K_V)_V = a - b \leq 0$. The effectivity of C implies $b > 0$, and the restriction morphism $\pi\big|_C : C \to \mathbf{P}^1$ is surjective. Let $\nu : D \to C$ be the normalization, and let $\mu = \pi \circ \nu : D \to \mathbf{P}^1$. The inclusion $D \to X = \mathbf{P}(\mathcal{E})$ determines the surjection $\mu^*\mathcal{E} = \nu^*(\pi^*\mathcal{E}) \to \nu^*\mathcal{O}(1) = \mathcal{O}_D(a)$. It follows from $\deg \mu = (C \cdot F_V)_V = b$ that $\mu^*\mathcal{O}_{\mathbf{P}^1}(1) = \mathcal{O}_D(b)$, and hence the above surjection gives the surjection $\mathcal{O}_D^{\oplus 2} \oplus \mathcal{O}_D(b)^{\oplus 4} \to \mathcal{O}_D(a)$. If $b > a$, then this surjection factors $\mathcal{O}_D^{\oplus 2} \to \mathcal{O}_D(a)$. It means that D is contained in $\mathbf{P}(\mathcal{O}^{\oplus 2}) \subset X = \mathbf{P}(\mathcal{E})$, but this cannot occur because $V \cap \mathbf{P}(\mathcal{O}^{\oplus 2}) = \emptyset$. Thus we conclude that $a = b$ and $(C \cdot -K_V)_V = 0$, in other words, $-K_V$ is nef.

Consequently, V is the weak Fano 3-fold with del Pezzo fibration of degree 5.

The anti-canonical model

As above, the weak Fano 3-fold V defined by (S) has the nef and big anti-canonical divisor $-K_V \sim H_V - F_V$ with $(-K_V)_V^3 = 2$. The anti-canonical morphism $\varphi_{|-K_V|} : V \to \mathbf{P}^3$ is obtained by restricting the morphism $\varphi_{|H-F|} : \mathbf{P}(\mathcal{E}) \to \mathbf{P}^3$. Let $V \to \overline{V} \to \mathbf{P}^3$ be the Stein factorization of $\varphi_{|-K_V|} : V \to \mathbf{P}^3$, $\overline{V}$ the anti-canonical model of V. It follows from $(-K_V)_V^3 = 2$ that the anti-canonical model $\overline{V} \to \mathbf{P}^3$ is a double covering.

In this section, we will describe the anti-canonical model $\overline{V} \to \mathbf{P}^3$ explicitly by the homogeneous coordinates $[y_2 : y_3 : y_4 : y_5]$ of $\mathbf{P}^3$. For the sake of convenience, we set

$$\Delta := y_2y_3 + {y_4}^2, \qquad A := ay_3 - by_2, \qquad B := cy_3 - dy_2,$$
$$\Phi := Ay_4 + By_3, \qquad \Psi := -Ay_2 + By_4,$$
$$H := (ad - bc)y_4 + Aa + Bd, \qquad G := B\Phi + A\Psi$$

and use the notation

$$\langle U, V\rangle := u_1v_1 - 2u_0v_2 - 2u_2v_0, \qquad \text{and} \qquad [pq] := p_0q_1 - p_1q_0$$

for $U = u_0{t_0}^2 + u_1t_0t_1 + u_2{t_1}^2$, $V = v_0{t_0}^2 + v_1t_0t_1 + v_2{t_1}^2$, $p = p_0t_0 + p_1t_1$ and $q = q_0t_0 + q_1t_1$. Then the defining equation of the branch locus Σ of $\overline{V} \to \mathbf{P}^3$ can be described by

$$D = \Delta^2\langle g, g\rangle - 2\Delta\langle g, H\rangle - 2\langle g, G\rangle + \langle H, H\rangle + 4([ad] - [bc])[AB].$$

Before confirming of this, we summarize the useful relations derived by the straightforward calculation:

Lemma 1. *Under the above setting and notation, we have*

(1) $y_4\Phi - y_3\Psi = A\Delta, \qquad y_2\Phi + y_4\Psi = B\Delta;$

(2) $Hy_2 = c\Phi - a\Psi - B^2, \qquad Hy_3 = d\Phi - b\Psi + A^2,$
$Hy_4 = a\Phi + d\Psi - AB + (ad - bc)\Delta;$

(3) $Gy_2 = -\Psi^2 + B^2\Delta, \qquad Gy_3 = \Phi^2 - A^2\Delta, \qquad Gy_4 = \Phi\Psi + AB\Delta.$

Lemma 2. *Let* $U = u_0{t_0}^2 + u_1t_0t_1 + u_2{t_1}^2$, $V = v_0{t_0}^2 + v_1t_0t_1 + v_2{t_1}^2$, $W = w_0{t_0}^2 + w_1t_0t_1 + w_2{t_1}^2$, $p = p_0t_0 + p_1t_1$, $q = q_0t_0 + q_1t_1$, $r = r_0t_0 + r_1t_1$ *and* $s = s_0t_0 + s_1t_1$. *Then*

(1) $\langle U, V\rangle = \langle V, U\rangle, \qquad \langle U + V, W\rangle = \langle U, W\rangle + \langle V, W\rangle;$

(2) $[pq] = -[qp], \qquad [(p + q)r] = [pr] + [qr],$ *in particular* $[pp] = 0;$

(3) $\langle pq, rs\rangle = -[pr][qs] - [ps][qr];$

(4) $\langle p^2, rs\rangle = -2[pr][ps], \qquad \langle pq, ps\rangle = [pq][ps].$

Lemma 3. *Under the above setting and notation, we have*

(1) $[A\Phi]=[AB]y_3,\quad [B\Phi]=-[AB]y_4,\quad [A\Psi]=[AB]y_4,\quad [B\Psi]=[AB]y_2;$
(2) $[\Phi\Psi]=[AB]\Delta.$

We now confirm the description for the branch locus, eliminating the variables x_i and t_j in the defining equations (S). From two equations

$$f_2=x_0y_4-(x_1+a)y_3+by_2=0 \quad \text{and}$$
$$f_1=x_1y_4-cy_3+(x_0+d)y_2=0,$$

we have

$$x_0\Delta=\Phi \quad \text{and} \quad x_1\Delta=\Psi.$$

Eliminating x_0 and x_1 in f_5 by them, we have

$$\begin{aligned} f_5\Delta^2 &= gy_2\Delta^2-\Phi c\Delta+\Psi(\Psi+a\Delta) \\ &= gy_2\Delta^2-(B^2+Hy_2)\Delta+(B^2\Delta-Gy_2) \qquad \text{(by Lemma 1)} \\ &= (g\Delta^2-H\Delta-G)y_2, \end{aligned}$$

and similarly we have

$$f_4\Delta^2=gy_3\Delta^2-\Phi(\Phi+d\Delta)+\Psi b\Delta=(g\Delta^2-H\Delta-G)y_3,$$
$$f_3\Delta^2=gy_2\Delta^2-(\Psi+a\Delta)(\Phi+d\Delta)+bc\Delta^2=(g\Delta^2-H\Delta-G)y_4$$

by Lemma 1. Thus we obtain the following equation if $(y_2,y_3,y_4)\neq(0,0,0)$:

$$F:=g\Delta^2-H\Delta-G=0.$$

(It holds $F=0$ trivially if $(y_2,y_3,y_4)=(0,0,0)$.) The polynomial F in y_i and t_j is written by

$$F=F_0{t_0}^2+F_1t_0t_1+F_2{t_1}^2, \qquad F_i=g_i\Delta^2-H_i\Delta-G_i \quad (i=0,1,2),$$

for the homogeneous polynominal F_i of degree 5 in $y_2,\cdots,y_5$. Since F is a quadratic form in t_j, the equation $F=0$ gives two values of $[t_0:t_1]$ for general $y=[y_2:y_3:y_4:y_5]\in\mathbf{P}^3$, i.e., the anti-canonical model $\overline{V}$ is a double covering of $\mathbf{P}^3$. The branch locus Σ is defined by

$$D:=({F_1}^2-4F_0F_2)/\Delta^2=0.$$

Here D is a polynominal of degree 6 in y_i's as follows. Since

$$\begin{aligned} &{F_1}^2-4F_0F_2 \\ &=(g_1\Delta^2-H_1\Delta-G_1)^2-4(g_0\Delta^2-H_0\Delta-G_0)(g_2\Delta^2-H_2\Delta-G_2) \\ &=\Delta^4({g_1}^2-4g_0g_2)-2\Delta^3(g_1H_1-2g_0H_2-2g_2H_0) \\ &\qquad -2\Delta^2(g_1G_1-2g_0G_2-2g_2G_0)+\Delta^2({H_1}^2-4H_0H_2) \\ &\qquad +2\Delta(H_1G_1-2H_0G_2-2H_2G_0)+({G_1}^2-4G_0G_2) \\ &=\Delta^4\langle g,g\rangle-2\Delta^3\langle g,H\rangle-2\Delta^2\langle g,G\rangle+\Delta^2\langle H,H\rangle+2\Delta\langle H,G\rangle+\langle G,G\rangle, \end{aligned}$$

we have only to show that

$$2\Delta\langle H,G\rangle+\langle G,G\rangle$$

has the factor Δ^2. Easy calculation shows that

$$\begin{cases} \langle G,G\rangle = 4[AB]^2\Delta, & \cdots\cdots\cdots \quad \text{(R1)} \\ \langle H,G\rangle = 2[AB]\Big(([ad]-[bc])\Delta-[AB]\Big). & \cdots\cdots\cdots \quad \text{(R2)} \end{cases}$$

Indeed, the relation (R1) is follows from

$$\begin{aligned} &\langle G,G\rangle = \langle B\Phi+A\Psi, B\Phi+A\Psi\rangle \\ &= \langle B\Phi,B\Phi\rangle + 2\langle B\Phi,A\Psi\rangle + \langle A\Psi,A\Psi\rangle \\ &= [B\Phi]^2 - 2[BA][\Phi\Psi] - 2[B\Psi][\Phi A] + [A\Psi]^2 && \text{(by Lemma 2)} \\ &= ([AB]y_4)^2 - 2[BA][AB]\Delta + 2[AB]^2y_2y_3 + ([AB]y_4)^2 && \text{(by Lemma 3)} \\ &= 4[AB]^2\Delta. \end{aligned}$$

It is more complicated for (R2). It follows from Lemmas 1, 2 and 3 that

$$\begin{aligned} &\langle H,G\rangle {y_2}^2 = \langle Hy_2, Gy_2\rangle = \langle c\Phi - a\Psi - B^2, -\Psi^2 + B^2\Delta\rangle \\ &= 2[c\Psi][\Phi\Psi] - 2[B\Psi]^2 - 2([cB][\Phi B] - [aB][\Psi B])\Delta \\ &= 2[c\Psi][AB]\Delta - 2[AB]^2{y_2}^2 - 2([cB][AB]y_4 + [aB][AB]y_2)\Delta \\ &= 2[AB](-[cA]y_2 + [cB]y_4 - [cB]y_4 - [aB]y_2)\Delta - 2[AB]^2{y_2}^2 \\ &= 2[AB](-[ca]y_3 + [cb]y_2 - [ac]y_3 + [ad]y_2)y_2\Delta - 2[AB]^2{y_2}^2 \\ &= 2[AB]\Big(([ad]-[bc])\Delta - [AB]\Big){y_2}^2 \end{aligned}$$

and similarly

$$\langle H,G\rangle {y_3}^2 = 2[AB]\Big(([ad]-[bc])\Delta - [AB]\Big){y_3}^2.$$

If $y_2 \neq 0$ or $y_3 \neq 0$, we obtain (R2). The rest is the case $y_2 = y_3 = 0$. In this case, $A = B = 0$ and the both side of (R2) are also 0, hence the equality (R2) holds. Thus we have

$$2\Delta\langle H,G\rangle + \langle G,G\rangle = 4([ad]-[bc])[AB]\Delta^2$$

by (R1) and (R2), and therefore ${F_1}^2 - 4F_0F_2$ is divided by Δ^2. Consequently, it follows that

$$\begin{aligned} D &= ({F_1}^2 - 4F_0F_2)/\Delta^2 \\ &= \Delta^2\langle g,g\rangle - 2\Delta\langle g,H\rangle - 2\langle g,G\rangle + \langle H,H\rangle + 4([ad]-[bc])[AB]. \end{aligned}$$

and hence the branch locus of the double covering $\overline{V} \to \mathbf{P}^3$ is the surface Σ of degree 6 defined by $D = 0$.

We next represent the anticanonical model $\overline{V}$ and the morphism $V \to \overline{V}$ by using the bihomogeneous coordinates rings. Let $R = \mathbf{C}[x_0, x_1, y_2, y_3, y_4, y_5, t_0, t_1]$ and $R_V = R/(f_1, \cdots, f_5)$ be the bihomogeneous coordinates rings of $X = \mathbf{P}(\mathcal{E})$ and $V \subset X$, respectively. The anticanonical model $\overline{V}$ is the double cover of $\mathbf{P}^3$ branched along $B = (D) \subset \mathbf{P}^3$, and hence is written by $w^2 - D$ in the weighted homogeneous coordinates ring $R_W = \mathbf{C}[y_2, y_3, y_4, y_5; w]$ with $\deg y_i = 1$ and $\deg w = 3$, i.e, $\overline{V} = (w^2 - D) \subset \mathbf{Proj}\ R_W = \mathbf{P}(1^4, 3)$. The ring homomorphisms $\varphi_i^{\#} : R_{\overline{V}} = R_W/(w^2 - D) \to R_V[\frac{1}{t_i}]$ defined by

$$\varphi_0^{\#}: \begin{array}{l} y_2 \mapsto y_2 \\ y_3 \mapsto y_3 \\ y_4 \mapsto y_4 \\ y_5 \mapsto y_5 \\ w \mapsto (F_1 t_0 + 2F_2 t_1)/(t_0 \Delta) \end{array} \qquad \varphi_1^{\#}: \begin{array}{l} y_2 \mapsto y_2 \\ y_3 \mapsto y_3 \\ y_4 \mapsto y_4 \\ y_5 \mapsto y_5 \\ w \mapsto -(2F_0 t_0 + F_1 t_1)/(t_1 \Delta) \end{array}$$

are glued over $R_V[\frac{1}{t_0}, \frac{1}{t_1}]$, becouse $(F_1t_0 + 2F_2t_1)t_1 + (2F_0t_0 + F_1t_1)t_0 = 2F = 0$ on V; therefore they induce the morphism $\varphi : V \to \overline{V} \subset \mathbf{P}(1^4, 3)$. (In the case $\Delta = 0$, we can see that $(F_1t_0 + 2F_2t_1)/t_0$ and $(2F_0t_0 + F_1t_1)/t_1$ are factored by Δ as in the after section constructing the flopped model.) This is an isomorphism outside of the locus (F_0, F_1, F_2) whose inverse image has positive dimension. The locus coincides with the set of flopping curves discussed in the next section.

The flopping curves

The morphism $\varphi_{|-K_V|} : V \to \mathbf{P}^3$ contracts the curves $C \subset V$ with $(C \cdot -K_V) = 0$. Since these curves are in fibers of $\varphi_{|-K_V|}$, in order to find the curves $C \subset V$ with $(C \cdot -K_V) = 0$, we have only to find points in $\mathbf{P}^3$ with positive dimensional fibers of $\varphi_{|-K_V|}$. Eliminating the variables x_0 and x_1 from the system (S) as in the previous section, we have the system of equations in the four variables y_2, y_3, y_4 and y_5. The solutions independent of the value $[t_0 : t_1]$ are related to the points with positive dimensional fibers. We treat into two cases.

Case (1). We treat the case $\Delta = 0$. In this case, there is the special subcase $y_2 = y_3 = y_4 = 0$ corresponding to the point $P = [0 : 0 : 0 : 1] \in \mathbf{P}^3$. The fiber γ over P is the curve defined by

$$\begin{cases} f_5 = -x_0 c^{(5)} y_5 + x_1(x_1 + a^{(5)} y_5) = 0 \\ f_4 = -x_0(x_0 + d^{(5)} y_5) + x_1 b^{(5)} y_5 = 0 \\ f_3 = -(x_1 + a^{(5)} y_5)(x_0 + d^{(5)} y_5) + b^{(5)} c^{(5)} {y_5}^2 = 0 \end{cases}$$

in the sub-$\mathbf{P}^2$-bundle $\mathbf{P}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)) \to \mathbf{P}^1$ of $\pi : X \to \mathbf{P}^1$. Here $a^{(5)}$, $b^{(5)}$, $c^{(5)}$ and $d^{(5)}$ are the coefficients of y_5 in the polynomials a, b, c and d, respectively. Generality of these polynomials implies that γ is irreducible. Since $\gamma \equiv 3l + 3s_0$ in the $\mathbf{P}^2$-bundle, hence in $V \subset X$, the curve γ is the trisection of $\pi : V \to \mathbf{P}^1$ with $(\gamma \cdot -K_V) = 0$.

Next consider the other case, i.e., the case $\Delta = 0$ but $y_2 \neq 0$ or $y_3 \neq 0$. As in the previous section, we have $\Delta = \Phi = \Psi = 0$ from $f_2 = f_1 = 0$, the system of equations in the variables y_i and t_j. We have only to find the values

y_i, independent of the ratio $[t_0 : t_1]$, satisfying the system. These values satisfy

$$\Delta = \Phi_0 = \Phi_1 = \Psi_0 = \Psi_1 = 0, \qquad \cdots\cdots\cdots \quad \text{(S3)}$$

where Φ_i, Ψ_i are the coefficients of t_i in Φ, Ψ. Let $\mathbf{Q} \subset \mathbf{P}^3$ be the singular quadric surface defined by $\Delta = 0$, and $\widetilde{\mathbf{Q}} \to \mathbf{Q}$ the minimal resolution of $\mathbf{Q}$. Then $\widetilde{\mathbf{Q}}$ is the Hirzebruch surface $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathbf{P}^1$ with the minimal section σ, $(\sigma^2)_{\widetilde{\mathbf{Q}}} = -2$. The above special point P is nothing but the vertex of $\mathbf{Q}$. Let $C_i \subset \mathbf{Q}$ be the intersection curve with the surface defined by $\Phi_i = \Psi_i = 0$, and let $\widetilde{C}_i \subset \widetilde{\mathbf{Q}}$ be the strict transforms of C_i. The curves C_0 and C_1 meet at $P \in \mathbf{Q}$, but the generality of the polynominals a, b, c, and d implies that $\widetilde{C}_0$ and $\widetilde{C}_1$ does not meet on the minimal section $\sigma \subset \widetilde{\mathbf{Q}}$. Locally calculation gives $\widetilde{C}_i \sim 3(2f+\sigma)-(f+2\sigma) = 5f+\sigma$ on $\widetilde{\mathbf{Q}}$, where f is a fiber of $\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(2)) \to \mathbf{P}^1$. We have $(\widetilde{C}_1 \cdot \widetilde{C}_2)_{\widetilde{\mathbf{Q}}} = ((5f+\sigma)^2)_{\widetilde{\mathbf{Q}}} = 10-2 = 8$, whence, except for the point $P \in \mathbf{Q} \subset \mathbf{P}^3$, there are 8 points on $\mathbf{Q}$ with positive dimensional fiber of $\varphi_{|-K_V|}$. The fiber over the point $\eta = [\eta_2 : \cdots : \eta_5]$ of them is the curve defined by

$$\begin{cases} f_5 = g(\eta)\eta_2 y^2 - x_0 c(\eta) y + x_1(x_1 + a(\eta)y) = 0 \\ f_2 = x_1\eta_4 - c(\eta)\eta_3 y + (x_0 + d(\eta)y)\eta_2 = 0 \end{cases}$$

in the sub-$\mathbf{P}^2$-bundle of $X = \mathbf{P}(\mathcal{E})$ corresponding to the value η, isomorphic to $\mathbf{P}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1))$ with the bihomogeneous coordinate system $[x_0 : x_1; y] \times [t_0 : t_1]$. Here $a(\eta), \cdots, g(\eta)$ mean the evaluations of $a, \cdots, g$ by $\eta = [\eta_2 : \cdots : \eta_5]$. (This representation of the fiber is in the case $\eta_2 \neq 0$. It is similar in the case $\eta_3 \neq 0$.) The curve is irreducible, because a, b, c, d and g are general (b appears in the case $\eta_3 \neq 0$). This is the bisection in $\mathbf{P}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)) \to \mathbf{P}^1$. Consequently, there are 8 bisections $\tau_1, \cdots, \tau_8$ of $\pi : V \to \mathbf{P}^1$, contracted by $\varphi_{|-K_V|}$.

Case (2). In the case $\Delta \neq 0$, we have the following equation in the variables y_i and t_j:

$$F = g\Delta^2 - H\Delta - G = 0$$

with $H = (ad-bc)y_4 + Aa + Bd$ and $G = B\Phi + A\Psi$. The values $y = [y_2 : y_3 : y_4 : y_5] \in \mathbf{P}^3$ independent of the ratio $[t_0 : t_1]$ are given by

$$F_0 = F_1 = F_2 = 0, \qquad \cdots\cdots\cdots \quad \text{(S4)}$$

where $F_i = g_i\Delta^2 - H_i\Delta - G_i$ is the coefficient of ${t_0}^i {t_1}^{2-i}$ in F. Since each F_i is the homogeneous polynomial of degree 5 in y_i, the system (S4) has 125 solutions counted with the multiplicity. We first show that each value found in Case (1) satisfies also (S4). Since F_i belongs to the ideal $(y_2, y_3, y_4)^3 \subset \mathbf{C}[y_2, y_3, y_4, y_5]$ but does not to $(y_2, y_3, y_4)^4$, the system (S4) has zero with multiplicity 27 at $P = [0 : 0 : 0 : 1]$. The solution $\eta = [\eta_2 : \cdot : \eta_5]$ of (S3) except P satisfies that $\eta_2 \neq 0$ or $\eta_3 \neq 0$. Lemma 1 implies that

$$\begin{aligned} Fy_2 &= gy_2\Delta^2 - (c\Phi - a\Psi - B^2)\Delta - (-\Psi^2 + B^2\Delta) \\ &= gy_2\Delta^2 - (c\Phi - a\Psi)\Delta + \Psi^2, \qquad \text{and} \\ Fy_3 &= gy_3\Delta^2 - (d\Phi - b\Psi + A^2)\Delta - (\Phi^2 - A^2\Delta) \\ &= gy_3\Delta^2 - (d\Phi - b\Psi)\Delta - \Phi^2. \end{aligned}$$

Since Δ, Φ and Ψ are simple zero at η, the polynomial F has zeros at η with multiplicity two for any $[t_0 : t_1]$. Hence the system (S4) has zero with multiplicity 8 at each η, the solution of (S3) except P. Therefore, the number of the solutions of (S4) counted with the multiplicity, which does not satisfy (S3), is $125 - 27 - 8 \times 8 = 34$. For the general coefficients of a, b, c, d and g, these 34 roots have the simple multiplicity. (For example, we can check the simplicity of these roots in the case $f = 2y_3{t_0}^2 + (y_2 - 3y_3 - 4y_4 - 9y_5)t_0t_1 - (y_3 + 2y_5){t_1}^2$, $a = (-y_2 + y_5)t_1$, $b = (y_2 + y_3 + 2y_4 + y_5)t_0 - y_5t_1$, $c = (-y_2 + y_3 + y_4 + y_5)t_0 + y_2t_1$ and $d = 2(y_3 + y_5)t_0 + y_3t_1$.) For each $\zeta = [\zeta_2 : \cdots : \zeta_5]$ of these simple roots, the fiber over $\zeta \in \mathbf{P}^3$ is the curve defined by

$$\begin{cases} f_2 = x_0\zeta_4 - (x_1 + a(\zeta)y)\zeta_3 + b(\zeta)\zeta_2 y = 0 \\ f_1 = x_1\zeta_4 - c(\zeta)\zeta_3 y + (x_0 + d(\zeta)y)\zeta_2 = 0 \end{cases}$$

in the sub-$\mathbf{P}^2$-bundle of X corresponding to ζ, isomorphic to $\mathbf{P}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1))$ with the bihomogeneous coordinate system $[x_0 : x_1; y] \times [t_0 : t_1]$. Here $a(\zeta), \cdots, d(\zeta)$ are the evaluations of $a, \cdots, d$ by $\zeta = [\zeta_2 : \cdots : \zeta_5]$. This curve is a section of $\pi : V \to \mathbf{P}^1$.

Summing up the argument above, one has that there exist 43 flopping curves on V, and that one of them is a trisection γ, 8 of them are bisections $\tau_1, \cdots, \tau_8$, and the rest are sections $\sigma_1, \cdots, \sigma_{34}$ of $\pi : V \to \mathbf{P}^1$.

The flopped model

We last construct the flop V^+ of the Fano 3-fold V concretely. The 3-fold $V \subset X = \mathbf{P}(\mathcal{E})$ is defined by the system of equations (S). The polynominal $F = g\Delta^2 - H\Delta - G$ eliminating x_0 and x_1 is of bidegree $(5, 2)$ in $\mathbf{C}[y_2, y_3, y_4, y_5, t_0, t_1]$, i.e., $F = F_0{t_0}^2 + F_1t_0t_1 + F_2{t_1}^2$.

Let $R^+ = \mathbf{C}[z_0, z_1, y_2, y_3, y_4, y_5, s_0, s_1]$ with $\deg z_i = (1, 0)$, $\deg y_i = (1, -1)$, $\deg s_j = (0, 1)$ be the bihomogeneous coordinates ring of the projective space bundle $X^+ = \mathbf{P}(\mathcal{E}^+)$, $\mathcal{E}^+ = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 4} \cong \mathcal{E}$. We consider the following system of equations (S$^+$):

$$\left\{\begin{aligned} f_5^+ &= g^+ y_2 - z_0 c^+ + z_1(z_1 + a^+) \\ f_4^+ &= g^+ y_3 - z_0(z_0 + d^+) + z_1 b^+ \\ f_3^+ &= g^+ y_4 - (z_1 + a^+)(z_0 + d^+) + b^+ c^+ \\ f_2^+ &= z_0 y_4 - (z_1 + a^+) y_3 + b^+ y_2 \\ f_1^+ &= z_1 y_4 - c^+ y_3 + (z_0 + d^+) y_2 \end{aligned}\right. \qquad \cdots\cdots\cdots \qquad (\mathrm{S}^+)$$

with

$$\left\{\begin{aligned} g^+ &= g_0 {s_0}^2 + g_1 s_0 s_1 + g_2 {s_1}^2 \\ a^+ &= a_0 s_0 + a_1 s_1 \\ b^+ &= b_0 s_0 + b_1 s_1 \\ c^+ &= c_0 s_0 + c_1 s_1 \\ d^+ &= d_0 s_0 + d_1 s_1 \end{aligned}\right.$$

and let $V^+ \subset X^+$ be the variety defined by (S^+), i.e., $V^+ = \mathbf{Proj}\, R^+/(f_1^+, \cdots, f_5^+)$. Let $\chi_j^\# : R^+/(f_1^+, \cdots, f_5^+) \to R/(f_1, \cdots, f_5)[\frac{1}{t_j}]$ $(j = 0, 1)$ be the ring homomorphism defined as follows:

$$\chi_0^\# : \begin{aligned} z_0 &\mapsto (x_0 F_2 + \varphi_0 t_0)/{t_0}^2 \\ z_1 &\mapsto (x_1 F_2 + \psi_0 t_0)/{t_0}^2 \\ y_2 &\mapsto y_2 \\ y_3 &\mapsto y_3 \\ y_4 &\mapsto y_4 \\ y_5 &\mapsto y_5 \\ s_0 &\mapsto -F_2/t_0 \\ s_1 &\mapsto (F_1 t_0 + F_2 t_1)/{t_0}^2 \end{aligned} \qquad \chi_1^\# : \begin{aligned} z_0 &\mapsto (x_0 F_0 + \varphi_1 t_1)/{t_1}^2 \\ z_1 &\mapsto (x_1 F_0 + \psi_1 t_1)/{t_1}^2 \\ y_2 &\mapsto y_2 \\ y_3 &\mapsto y_3 \\ y_4 &\mapsto y_4 \\ y_5 &\mapsto y_5 \\ s_0 &\mapsto (F_0 t_0 + F_1 t_1)/{t_1}^2 \\ s_1 &\mapsto -F_0/t_1 \end{aligned}$$

Here

$$\begin{aligned} \varphi_0 &= (-2\Phi_0 g_2 + \Phi_1 g_1)\Delta - (-2\Phi_0 H_2 + \Phi_1 H_1) + 2[AB]A_1, \\ \psi_0 &= (-2\Psi_0 g_2 + \Psi_1 g_1)\Delta - (-2\Psi_0 H_2 + \Psi_1 H_1) - 2[AB]B_1, \\ \varphi_1 &= (-2\Phi_1 g_0 + \Phi_0 g_1)\Delta - (-2\Phi_1 H_0 + \Phi_0 H_1) - 2[AB]A_0, \\ \psi_1 &= (-2\Psi_1 g_0 + \Psi_0 g_1)\Delta - (-2\Psi_1 H_0 + \Psi_0 H_1) + 2[AB]B_0. \end{aligned}$$

Since

$$\begin{aligned} & 2\Phi_0 G_2 - \Phi_1 G_1 \\ &= 2\Phi_0(A_1\Psi_1 + B_1\Phi_1) - \Phi_1(A_0\Psi_1 + A_1\Psi_0 + B_0\Phi_1 + B_1\Phi_0) \\ &= [\Phi A]\Psi_1 + [\Phi\Psi]A_1 + [\Phi B]\Phi_1 \\ &= -[AB]\Psi_1 y_3 + [AB]\Delta A_1 + [AB]\Phi_1 y_4 \\ &= 2[AB]A_1\Delta \end{aligned}$$

and similarly

$$2\Phi_1 G_0 - \Phi_0 G_1 = -2[AB]A_0\Delta$$

by Lemmas 1 and 3, we have

$$
\begin{aligned}
&(x_0F_2+\varphi_0t_0)\Delta{t_1}^2\\
&=\Phi F_2{t_1}^2+\varphi_0\Delta t_0{t_1}^2\\
&=\Phi F_2{t_1}^2+(-2\Phi_0(F_2+G_2)+\Phi_1(F_1+G_1)+2[AB]A_1\Delta)t_0{t_1}^2\\
&=((\Phi_0t_0+\Phi_1t_1)F_2-2\Phi_0F_2t_0+\Phi_1F_1t_0){t_1}^2-(2\Phi_0G_2-\Phi_1G_1-2[AB]A_1\Delta)t_0{t_1}^2\\
&=(-\Phi_0F_2t_0t_1+\Phi_1(F_1t_0t_1+F_2{t_1}^2))t_1\\
&=-(\Phi_0F_2t_1+\Phi_1F_0t_0)t_0t_1
\end{aligned}
$$

and similarly

$$(x_0F_0+\varphi_1t_1)\Delta{t_0}^2=-(\Phi_0F_2t_1+\Phi_1F_0t_0)t_0t_1$$

modulo $(f_1,\cdots,f_5)$. The similar calculation also gives $(x_1F_2+\psi_0t_0)\Delta{t_1}^2 = (x_1F_0+\psi_1t_1)\Delta{t_0}^2$ modulo $(f_1,\cdots,f_5)$, therefore the ring homomorphisms $\chi_0^{\#}$ and $\chi_1^{\#}$ glue over $R/(f_1,\cdots,f_5)[\frac{1}{t_0},\frac{1}{t_1}]$, and induce the rational map $\chi: V\cdots\to V^+$. This map is an isomorphism outside of $F_0=F_1=F_2=0$ which is nothing but the set of flopping curves.

By symmetry, the variety V^+ is a smooth weak Fano 3-fold with del Pezzo fibration of degree 5: $V^+\subset\mathbf{P}(\mathcal{E}^+)\to\mathbf{P}^1$, and has the parallel properties to the original weak Fano 3-fold $V\subset\mathbf{P}(\mathcal{E})\to\mathbf{P}^1$. In particular, the anticanonical model is the same to $\overline{V}$, and the flop $\chi: V\cdots\to V^+$ is obtained by extending the involution map of the double cover $\overline{V}\to\mathbf{P}^3$.

Result

Consequently, we have the following diagram of varieties and birational maps:

$$
\begin{tikzcd}[column sep=small]
& & \{\gamma,\tau_1,\cdots,\tau_8,\sigma_1,\cdots,\sigma_{34}\} : \text{flopping curves} \arrow[d, equal] & & & & \\
\mathbf{F}(0^2,1^4)=X \arrow[dd, "\pi"'] & \supset V \arrow[ddrr] & \supset C \arrow[rr, dashed, leftrightarrow, "\chi", "\text{flop}"'] \arrow[dr, "\varphi"'] & & C^+ \arrow[dl, "\varphi^+"] & \subset V^+ \arrow[ddll] & \subset X^+=\mathbf{F}(0^2,1^4) \arrow[dd, "\pi^+"] \\
& & & \overline{V} \arrow[d, "2:1"'] & & & \\
\mathbf{P}^1 & & & \mathbf{P}^3\supset B : \text{branch locus} & & & \mathbf{P}^1
\end{tikzcd}
$$

In this diagram, V is the weak Fano 3-fold with del Pezzo fibration of degree 5 of $(-K_V)^3=2$; the variety $\overline{V}$ is the anti-canonical model of V, which is the double cover of $\mathbf{P}^3$ branched sextic surface $B=(D)$; the subset $C\subset V$ is the union of flopping curves which consists of a trisection, 8 bisections and 34 sections of $\pi: V\to\mathbf{P}^1$; and the flop V^+ of V is also the weak Fano 3-fold with the same property to V.

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■註

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