原著（*Article*）

Maillet Type Theorem for Nonlinear Goursat Problems

非線形グルサー問題に対するマイエ型定理

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Abstract

Let $(t,x) \in \mathbb{C}^2$. The following equation is called the nonlinear Goursat problems.

$$\text{(E)} \qquad \begin{cases} \partial_t^K \partial_x^L u(t,x) = a(x)t^{K_0} + f_{K_0+1}\left(t, x, \left\{\partial_t^k \partial_x^\ell u(t,x)\right\}_\Delta\right), \\ u(t,x) = O(t^{K_0+K}), \\ u(t,x) - \varphi(t,x) = O(t^{K_0+K}x^L), \end{cases}$$

where $\varphi(x) = O(t^{K_0+K})$ is holomorphic in a neighborhood of the origin. The other definitions of notations will be stated later. For linear Goursat problems, Miyake [2], Miyake and Hashimoto [3] studied the solvability of solutions on the Gevrey spaces.

The purpose of this paper is to give the Maillet type theorem for nonlinear Goursat problems.

Keywords. Partial differential equations, Goursat problems, Maillet type theorem

1. Main Theorem

Let $(t,x) \in \mathbb{C}^2$. We consider the following Goursat problem for nonlinear partial differential equation.

$$\text{(E)} \qquad \begin{cases} \partial_t^K \partial_x^L u(t,x) = a(x)t^{K_0} + f_{K_0+1}\left(t, x, \left\{\partial_t^k \partial_x^\ell u(t,x)\right\}_\Delta\right), \\ u(t,x) = O(t^{K_0+K}), \\ u(t,x) - \varphi(t,x) = O(t^{K_0+K}x^L). \end{cases}$$

Here $\varphi(t,x)$ denotes an arbitrary holomorphic function whose vanishing order in t is K_0+K where $K_0 = \max\{0, K_1 - K\} + 1 (\geq 1)$. We assume that K and L are nonnegative integers, and we put

$$\text{(1.1)} \qquad \Delta = \{(k,\ell); 0 \leq k \leq K_1, 0 \leq \ell \leq L_1\},$$

where K_1 and L_1 are nonnegative integers. Moreover we assume that $a(x)$ is holomorphic in a neighborhood of the origin and $f_{K_0+1}(t,x,\xi)$ ($\xi = \{\xi_{k\ell}\}_\Delta = \{\xi_{k\ell}\}_{(k,\ell)\in\Delta}$) is also holomorphic in a neighborhood of the origin with Taylor expansion

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$$f_{K_0+1}(t,x,\xi) = \sum_{V(p,\alpha)\geq K_0+1} f_{p\alpha}(x)t^p \prod_{\Delta} \xi_{k\ell}^{\alpha_{k\ell}},$$

where

$$V(p,\alpha) = p + \sum_{\Delta}(K_0+K-k)\alpha_{k\ell}, \tag{1.2}$$

and $\prod_{\Delta} = \prod_{k=0}^{K_1}\prod_{\ell=0}^{L_1}$ and $\sum_{\Delta} = \sum_{k=0}^{K_1}\sum_{\ell=0}^{L_1}$.

Then the following theorem holds.

Theorem 1 *The formal solution of the equation* (E) *exists uniquely, and it belongs to the Gevrey class of order at most $s+1$, where*

$$s = \max_{p,\alpha}\left\{\frac{M(p,\alpha)-(K+L)}{V(p,\alpha)-K_0}, 0\right\}, \tag{1.3}$$

and

$$M(p,\alpha) = \max\{k+\ell; \alpha_{k\ell}\neq 0, f_{p\alpha}(x)\not\equiv 0\}. \tag{1.4}$$

This means that the power series $\sum_{i\geq K_0+K} u_i(x)t^i/i!^s$ converges in a neighborhood of the origin for the formal solution $u(t,x)=\sum_{i\geq K_0+K} u_i(x)t^i$.

2. Newton Polygons

For the point $(a,b)\in\mathbb{R}^2$, we define the region $\Lambda_{(a,b)}$ by

$$\Lambda_{(a,b)} = \{(X,Y); X\leq a, Y\geq b\}\subset\mathbb{R}^2.$$

Let $u(t,x)=O(t^{K_0+K})$. For the left hand side $\partial_t^K\partial_x^L u(t,x)$ of (E) and each term

$$f_{p\alpha}(x)t^p\prod_{\Delta}\left(\partial_t^k\partial_x^\ell u(t,x)\right)^{\alpha_{k\ell}}$$

of Taylor expansion of $f_{K_0+1}(t,x,\{\partial_t^k\partial_x^\ell u(t,x)\}_\Delta)$, we define the points in $\mathbb{R}^2$ by

$$\partial_t^k\partial_x^\ell u(t,x) \leftrightarrow (K+L,K_0), \quad f_{p\alpha}(x)t^p\prod_{\Delta}\left(\partial_t^k\partial_x^\ell u(t,x)\right)^{\alpha_{k\ell}} \leftrightarrow (M(p,\alpha),V(p,\alpha)).$$

Then the Newton polygon $\mathcal{N}(\mathrm{E})$ is defined as follows.

$$\mathcal{N}(\mathrm{E}) = \mathrm{Ch}\left(\Lambda_{(K+L,K_0)}\bigcup\left(\bigcup\nolimits_{p,\alpha}\lambda_{(M(p,\alpha),V(p,\alpha))}\right)\right),$$

where $\mathrm{Ch}(\cdots)$ denotes the convex hull of $\{\cdots\}$ in $\mathbb{R}^2$.

The following theorem holds.

Theorem 2 *Let σ be the least positive slope of Newton polygon $\mathcal{N}$(E). Then the Gevrey order $s+1$ of the formal solution of* (E) *is given by $s=1/\sigma$.*

The proof of Theorem 2 is obtained by Theorem 1, immediately.

3. Proof of Theorem 1

We put $u(t,x)=\varphi(t,x)+v(t,x)$ $(v(t,x)=O(t^{K_0+K}x^L))$. By substituting this into the equation (E), we see that $v(t,x)$ satisfies the following equation.

$$(\mathrm{E}_1)\qquad \begin{cases} \partial_t^K\partial_x^L v(t,x) = -\varphi_{KL}(t,x)+a(x)t^{K_0} \\ \qquad\qquad +f_{K_0+1}\left(t,x,\{\varphi_{k\ell}(t,x)+\partial_t^k\partial_x^\ell v(t,x)\}_\Delta\right), \\ v(t,x)=O(t^{K_0+K}x^L), \end{cases}$$

where

$$\varphi_{kl}(t,x):=\partial_t^k\partial_x^\ell\varphi(t,x).$$

We know that all $\varphi_{k\ell}(t,x)$ are holomorphic in a neighborhood of the origin. Especially, $\varphi_{k\ell}(t,x)=O(t^{K_0+K-k})$ for all k and ℓ.

We put $\varphi_{k\ell}(t,x)=\psi_{k\ell}(x)t^{K_0+K-k}+\tilde{\varphi}_{k\ell}(t,x)$, where $\tilde{\varphi}_{k\ell}(t,x)=O(t^{K_0+K-k+1})$, $\tilde{a}(x)=a(x)-\psi_{KL}(t,x)$ and $\tilde{f}_{K_0+1}(t,x,\{\partial_t^k\partial_x^\ell u\}_\Delta)=f_{K_0+1}(t,x,\{\partial_t^k\partial_x^\ell u\}_\Delta)-\tilde{\varphi}_{KL}(t,x)=O(t^{K_0+1})$. Then the equation is reduced to the following.

$$\partial_t^K\partial_x^L v(t,x)=\tilde{a}(x)t^{K_0}+\tilde{f}_{K_0+1}\left(t,x,\left\{\varphi_{k\ell}(t,x)+\partial_t^k\partial_x^\ell v(t,x)\right\}_\Delta\right).$$

Here $\tilde{f}_{K_0+1}$ is rewritten as follows.

$$\begin{aligned}
&\tilde{f}_{K_0+1}\left(t,x,\left\{\varphi_{k\ell}(t,x)+\partial_t^k\partial_x^\ell v\right\}_\Delta\right) \\
&= \tilde{f}_{K_0+1}\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right)+\sum_{|\alpha|\ge 1}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}\tilde{f}_{K_0+1}}{\partial\xi^\alpha}\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right)(\varphi_{k\ell}(t,x))^{\alpha_{k\ell}} \\
&= \tilde{f}_{K_0+1}\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right) \\
&\quad+\sum_{|\alpha|\ge 1}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}\tilde{f}_{K_0+1}}{\partial\xi^\alpha}\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right)\left(\psi_{k\ell}(x)t^{K_0+K-k}\right)^{\alpha_{k\ell}} \\
&\quad+\sum_{|\alpha|\ge 1}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}\tilde{f}_{K_0+1}}{\partial\xi^\alpha}\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right)\left\{\varphi_{k\ell}(t,x)^{\alpha_{k\ell}}-\left(\psi_{k\ell}(x)t^{K_0+K-k}\right)^{\alpha_{k\ell}}\right\} \\
&=: \tilde{f}_{K_0+1}\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right)+f_1\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right)+f_2\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right).
\end{aligned}$$

We can easily see that the vanishing orders of f_1 and f_2 are K_0+1 and K_0+2, respectively. Therefore, we can put the rightmost side of above by $g_{K_0+1}(t,x,\{\partial_t^k\partial_x^\ell v\}_\Delta)$, where

$$g_{K_0+1}\left(t,x,\left\{\partial_t^k\partial_x^\ell v\right\}_\Delta\right)=\sum_{V(p,\alpha)\ge K_0+1}g_{p\alpha}(x)t^p\prod_\Delta\left(\partial_t^k\partial_x^\ell v\right)^{\alpha_{k\ell}},$$

$$V(p,\alpha)=p=\sum_\Delta(K_0+K-k)\alpha_{k\ell}\qquad\text{(same form as (1.2))}.$$

We remark that the vanishing order $V(p,\alpha)$ of each term of f_1 is the same representation as the original $V(p,\alpha)$, but (p,α) is different from the original (p,α). However, the Gevrey order is not change.

In this case, (E_1) is rewritten as follows.

$$(\mathrm{E}_1')\qquad \begin{cases} \partial_t^K\partial_x^L v(t,x)=\tilde{a}(x)t^{K_0}+g_{K_0+1}\left(t,x,\left\{\partial_t^k\partial_x^\ell v(t,x)\right\}_\Delta\right), \\ v(t,x)=O(t^{K_0+K}x^L). \end{cases}$$

We put $V(t,x)=\partial_t^K\partial_x^L v(t,x)$ as a new unknown function. This implies that $v(t,x)=\partial_t^{-K}\partial_x^{-L}V(t,x)$. Then ($\mathrm{E}_1'$) is reduced to the following.

$$(\mathrm{E}_2)\qquad \begin{cases} V(t,x)=\tilde{a}(x)t^{K_0}+g_{K_0+1}\left(t,x,\left\{\partial_t^{k-K}\partial_x^{\ell-L}V(t,x)\right\}_{\Delta}\right), \\ V(t,x)=O(t^{K_0}). \end{cases}$$

We consider the following equation.

$$(\mathrm{E}_3)\qquad W(t,x)=\frac{At^{K_0}}{(R-x)^{K_0+K+L+1}}+G_{K_0+1}\left(t,x,\left\{\partial_t^{k-K}\partial_x^{\ell-L}W\right\}_{\Delta}\right)$$

with $W(t,x)=O(t^{K_0})$, where $\tilde{a}(x)\ll A/(R-x)^{K_0+K+L+1}$ and

$$G_{K_0+1}(t,x,\xi):=\sum_{V(p,\alpha)\geq K_0+1}\frac{G_{p\alpha}}{(R-x)^{p+K_0|\alpha|+K+L+1}}t^p\prod_{\Delta}\xi_{k\ell}^{\alpha_{k\ell}}\gg g_{K_0+1}(t,x,\xi).$$

By the construction of (E_3), we obtain $V(t,x)\ll W(t,x)$.
For (E_3), the following proposition holds.

Proposition 1 *The equation* (E_3) *has a unique formal solution, and it belongs to the Gevrey class of order at most $s+1$. Here the constant s is same as* (1.3).

If we admit Proposition 1, then the proof of Theorem 1 is obtained immediately. Thus, the proof of Theorem 1 is completed. □

4. Proof of Proposition 1

We put $W(t,x)=\sum_{i\geq K_0}W_i(x)t^i$, and substituting this into (E_3), we have

$$\begin{aligned}\sum_{i\geq K_0}W_i(x)t^i &= \frac{At^{K_0}}{(R-x)^{K_0+K+L+1}} \\ &\quad+\sum_{V(p,\beta,\gamma,\delta,\zeta)\geq K_0+1}\frac{G_{p\beta\gamma\delta\zeta}}{(R-x)^{p+K_0(|\beta|+|\gamma|+|\delta|+|\zeta|)+K+L+1}}t^p \\ &\qquad\times\prod_{\Delta_1}\left(\sum_{i\geq K_0}\partial_x^{-(L-\ell)}W_i(x)\frac{t^{i+K-k}}{\prod_{q=1}^{K-k}(i+q)}\right)^{\beta_{k\ell}} \\ &\qquad\times\prod_{\Delta_2}\left(\sum_{i\geq K_0}\partial_x^{-(L-\ell)}W_i(x)\left(\prod_{q=1}^{k-K}(i+1-q)\right)t^{i+K-k}\right)^{\gamma_{k\ell}} \\ &\qquad\times\prod_{\Delta_3}\left(\sum_{i\geq K_0}\partial_x^{\ell-L}W_i(x)\frac{t^{i+K-k}}{\prod_{q=1}^{K-k}(i+q)}\right)^{\beta_{k\ell}} \\ &\qquad\times\prod_{\Delta_4}\left(\sum_{i\geq K_0}\partial_x^{\ell-L}W_i(x)\left(\prod_{q=1}^{k-K}(i+1-q)\right)t^{i+K-k}\right)^{\zeta_{k\ell}},\end{aligned}$$

where $\begin{cases}\Delta_1=\{(k,\ell);k\leq K,\ell\leq L\}, & \Delta_2=\{(k,\ell);k>K,\ell\leq L\}, \\ \Delta_3=\{(k,\ell);k\leq K,\ell>L\}, & \Delta_4=\{(k,\ell);k>K,\ell>L\},\end{cases}$ and $\Delta=\Delta_1\cup\Delta_2\cup\Delta_3\cup\Delta_4$,

$$\begin{aligned} V(p,\beta,\gamma,\delta,\zeta) &= p+\sum_{\Delta_1}(K_0+K-k)\beta_{k\ell}+\sum_{\Delta_2}(K_0+K-k)\gamma_{k\ell} \\ &\quad +\sum_{\Delta_3}(K_0+K-k)\delta_{k\ell}+\sum_{\Delta_4}(K_0+K-k)\zeta_{k\ell}. \end{aligned}$$

This is equivalent to

$$V(p,\alpha)=p+\sum_{\Delta}(K_0+K-k)\alpha_{k\ell}.$$

We obtain the following recurrence formula. For $i=K_0$,

$$W_{K_0}(x)=\frac{A}{(R-x)^{K_0+K+L+1}},$$

and for $i\geq K_0+1$,

$$\begin{aligned} W_i(x) &= \sum_{V(p,\beta,\gamma,\delta,\zeta)\geq K_0+1}\frac{G_{p\beta\gamma\delta\zeta}}{(R-x)^{p+K_0(|\beta|+|\gamma|+|\delta|+|\zeta|)+K+L+1}} \\ &\quad \times\sum_{(*)}\left\{\prod_{\Delta_1}\prod_{r=1}^{\beta_{k\ell}}\frac{\partial_x^{\ell-L}W_{i_{k\ell r}}(x)}{\prod_{q=1}^{K-k}(i_{k\ell r}+q)}\prod_{\Delta_2}\prod_{r=1}^{\gamma_{k\ell}}\prod_{q=1}^{k-K}(j_{k\ell r}+1-q)\cdot\partial_x^{\ell-L}W_{j_{k\ell r}}(x)\right. \\ &\quad \left.\times\prod_{\Delta_3}\prod_{r=1}^{\delta_{k\ell}}\frac{\partial_x^{\ell-L}W_{\tau_{k\ell r}}(x)}{\prod_{q=1}^{K-k}(\tau_{k\ell r}+q)}\prod_{\Delta_4}\prod_{r=1}^{\zeta_{k\ell}}\prod_{q=1}^{k-K}(\kappa_{k\ell r}+1-q)\cdot\partial_x^{\ell-L}W_{\kappa_{k\ell r}}(x)\right\}, \end{aligned}$$

where $\sum_{(*)}$ is taken over

$$\begin{aligned} & p+\sum_{\Delta_1}\sum_{r=1}^{\beta_{k\ell}}(i_{k\ell r}+K-k)+\sum_{\Delta_2}\sum_{r=1}^{\beta_{k\ell}}(j_{k\ell r}+K-k) \\ &\quad +\sum_{\Delta_3}\sum_{r=1}^{\delta_{k\ell}}(\tau_{k\ell r}+K-k)+\sum_{\Delta_4}\sum_{r=1}^{\zeta_{k\ell}}(\kappa_{k\ell r}+K-k)=i. \end{aligned}$$

Of course, this is also equivalent to

$$p+\sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(i_{k\ell r}+K-k)=i.$$

Lemma 1 *For $i\geq K_0$, the coefficent $W_i(x)$ is written by*

$$W_i(x)=\sum_{J=i}^{Mi-(M-1)K_0}\frac{W_{iJ}}{(R-x)^{J+K+L+1}}, \tag{4.5}$$

where $M=(K+L+2K_0)(K_0+1)+1(\geq K+L+2K_0+1)$ and $W_{iJ}\geq 0$.

By Lemma 1, we have the following majorant relations.

Lemma 2 (i) *The case that* $(k,\ell)\in\Delta_1$,

$$\partial_t^{k-K}\partial_x^{\ell-L}W(t,x)\ll\frac{t^{K-k}}{(R-x)^{\ell-L}}W(t,x).$$

(ii) *The case that* $(k,\ell)\in\Delta_2$,

$$\partial_t^{k-K}\partial_x^{\ell-L}W(t,x)\ll\begin{cases}\dfrac{t^{K-k}}{(R-x)^{\ell-L}}W(t,x) & (\textit{if } k+\ell\le K+L),\\[2mm] \dfrac{t^{K-k}}{(R-x)^{\ell-L}}(Mt\partial_t+L_1)^{k+\ell-K-L}W(t,x) & (\textit{if } k+\ell> K+L).\end{cases}$$

(iii) *The case that* $(k,\ell)\in\Delta_3$,

$$\partial_t^{k-K}\partial_x^{\ell-L}W(t,x)\ll\begin{cases}\dfrac{(\tilde{M}t)^{K-k}}{(R-x)^{\ell-L}}W(t,x) & (\textit{if } k+\ell\le K+L),\\[2mm] \dfrac{(\tilde{M}t)^{K-k}}{(R-x)^{\ell-L}}(Mt\partial_t+L_1)^{k+\ell-K-L}W(t,x) & (\textit{if } k+\ell> K+L),\end{cases}$$

where $\tilde{M}=M+L_1$.
(iv) *The case that* $(k,\ell)\in\Delta_4$,

$$\partial_t^{k-K}\partial_x^{\ell-L}W(t,x)\ll\frac{t^{K-k}}{(R-x)^{\ell-L}}(Mt\partial_t+L_1)^{k+\ell-K-L}W(t,x).$$

We accept Lemma 1 and 2, and continue the proof of Proposition 1.
We consider the following equation.

$$(\mathrm{E}_4)\qquad\begin{cases}Y(t,x)=\dfrac{At^{K_0}}{(R-x)^{K_0+K+L+1}}\\ \qquad\qquad +G_{K_0+1}\left(t,x,\left\{\dfrac{t^{K-k}Y}{(R-x)^{\ell-L}}\right\}_{\Delta_1},\left\{\dfrac{t^{K-k}\mathcal{L}_2Y}{(R-x)^{\ell-L}}\right\}_{\Delta_2},\right.\\ \qquad\qquad\qquad\left.\left\{\dfrac{(\tilde{M}t)^{K-k}\mathcal{L}_2Y}{(R-x)^{\ell-L}}\right\}_{\Delta_3},\left\{\dfrac{t^{K-k}\mathcal{L}_1Y}{(R-x)^{\ell-L}}\right\}_{\Delta_4}\right),\\ Y(t,x)=O(t^{K_0}).\end{cases}$$

where

$$\mathcal{L}_1=(Mt\partial_t+L_1)^{k+\ell-K-L}\quad\text{and}\quad\mathcal{L}_2=\begin{cases}1 & (k+\ell\le K+L),\\ \mathcal{L}_1 & (k+\ell>K+L).\end{cases}$$

The equation (E_4) is a singular ordinary differential equation in t with parameter x, and we know that the Gevrey order of formal solution is given by $s+1$, where

$$s=\max_{p,\alpha}\left\{\frac{M(p,\alpha)-(K+L)}{V(p,\alpha)-K_0},0\right\}.$$

This result is, for example, found in [1] or [4].
Thus, we obtain Proposition 1. $\square$

5. Proof of Lemma 1

In order to prove Lemma 1, it is sufficient to estimate the lower and the upper bound estimates of the power J of $1/(R-x)$ by induction.

First, the lower bound estimate is given as follows.

$$\begin{aligned}
J &\geq p + K_0|\alpha| + K + L + 1 + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(i_{k\ell r} + K + L + 1) \\
&= p + K + L + 1 + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(i_{k\ell r} + K + L + K_0 + 1) \\
&= \left(p + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(i_{k\ell r} + K - k)\right) + K + L + 1 + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(K_0 + L + 1 + k) \\
&= i + K + L + 1 + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(K_0 + L + 1 + k) \\
&\geq i + K + L + 1.
\end{aligned}$$

Next, the upper bound estimate is given as follows.

$$\begin{aligned}
J &\leq p + K_0|\alpha| + K + L + 1 + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(Mi_{k\ell r} - (M-1)K_0 + K + L + 1) \\
&= M\left(p + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(i_{k\ell r} + K - k)\right) - (M-1)\left(p + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(K_0 + K - k)\right) \\
&\quad + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(k - K) + \sum_{\Delta}\sum_{r=1}^{\alpha_{k\ell}}(K + L + K_0 + 1) + K + L + 1 \\
&\leq Mi - (M-1)V(p,\alpha) + (K_0 - 1)|\alpha| + (K + L + K_0 + 1)|\alpha| + K + L + 1 \\
&= Mi - (M-1)V(p,\alpha) + (K + L + 2K_0)|\alpha| + K + L + 1 \\
&\leq Mi - (M-1)V(p,\alpha) + (K + L + 2K_0)V(p,\alpha) + K + L + 1 \\
&= Mi - (M - K - L - 2K_0 - 1)V(p,\alpha) + K + L + 1 \\
&\leq Mi - (M - K - L - 2K_0 - 1)(K_0 + 1) + K + L + 1 \\
&= Mi - (M-1)K_0 - \{M - (K + L + 2K_0)(K_0 + 1) - 1\} + K + L + 1 \\
&= Mi - (M-1)K_0 + K + L + 1.
\end{aligned}$$

Thus, we obtain Lemma 1. □

6. Proof of Lemma 2

In the case (i), $k \leq K$ and $\ell \leq L$ hold. Then we have

$$\begin{aligned}
&\partial_t^{k-K}\partial_x^{\ell-L}W(t,x) \\
&= \partial_t^{k-K}\partial_x^{\ell-L}\sum_{i\geq K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{W_{iJ}}{(R-x)^{J+K+L+1}}t^i \\
&= \sum_{i\geq K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{C_1 W_{iJ}}{(R-x)^{J+K+L+1+(\ell-L)}}t^{i+K-k} \ll \frac{t^{K-k}}{(R-x)^{\ell-L}}W(t,x),
\end{aligned}$$

because

$$C_1 := \frac{1}{\prod_{q=1}^{K-k}(i+q)\prod_{q=1}^{L-\ell}(J+K+L+q)} \le 1.$$

Hence, we obtain Lemma 2 (i).

In the case (ii), $k > K$ and $\ell \le L$ hold. Then we have

$$\begin{aligned}
&\partial_t^{k-K}\partial_x^{\ell-L}W(t,x) \\
&= \partial_t^{k-K}\partial_x^{\ell-L}\sum_{i\ge K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{W_{iJ}}{(R-x)^{J+K+L+1}}t^i \\
&= \sum_{i\ge K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{\prod_{q=1}^{k-K}(i+K-k+q)}{\prod_{q=1}^{L-\ell}(J+K+L+q)}\frac{W_{iJ}}{(R-x)^{J+K+L+1+(\ell-L)}}t^{i+K-k} \\
&\ll \sum_{i\ge K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{i^{k-K}}{(J+K+L+1)^{L-\ell}}\frac{W_{iJ}}{(R-x)^{J+K+L+1+(\ell-L)}}t^{i+K-k}.
\end{aligned}$$

Here, we put $\mathcal{L}_1 = (Mt\partial_t + L_1)^{k+\ell-K-L}$. Since $\frac{i}{J+K+L+1} \le \frac{i}{i+K+L+1} \le 1 \le i$, the majorant relation

$$\begin{aligned}
\frac{i^{k-K}}{(J+K+L+1)^{L-\ell}} &\ll \begin{cases} 1 & (\text{if } \ k+\ell \le K+L), \\ (t\partial_t)^{k+\ell-K-L} & (\text{if } \ k+\ell > K+L) \end{cases} \\
&\ll \begin{cases} 1 & (\text{if } \ k+\ell \le K+L), \\ \mathcal{L}_1 & (\text{if } \ k+\ell > K+L) \end{cases} \\
&= \mathcal{L}_2
\end{aligned}$$

holds in the sense of operator for t^i. This is Lemma 2 (ii).

In the case (iii), $k \le K$ and $\ell > L$ hold. Then we have

$$\begin{aligned}
&\partial_t^{k-K}\partial_x^{\ell-L}W(t,x) \\
&= \partial_t^{k-K}\partial_x^{\ell-L}\sum_{i\ge K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{W_{iJ}}{(R-x)^{J+K+L+1}}t^i \\
&= \sum_{i\ge K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{\prod_{q=1}^{\ell-L}(J+L+q)}{\prod_{q=1}^{K-k}(i+q)}\frac{W_{iJ}}{(R-x)^{J+K+L+1+(\ell-L)}}t^{i+K-k} \\
&\ll \sum_{i\ge K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{(J+\ell)^{\ell-L}}{(i+1)^{K-k}}\frac{W_{iJ}}{(R-x)^{J+K+L+1+(\ell-L)}}t^{i+K-k}.
\end{aligned}$$

Here, by inequalities

$$\frac{J+\ell}{i+1} \le \frac{Mi-(M-1)K_0+L_1}{i+1} \le M + L_1 =: \tilde{M} \quad \text{and} \quad J+\ell \le Mi + L_1,$$

the majorant relation

$$\frac{(J+\ell)^{\ell-L}}{(i+1)^{K-k}} \ll \tilde{M}^{K-k} \times \begin{cases} 1 & (\text{if } \ k+\ell \le K+L), \\ \mathcal{L}_1 & (\text{if } \ k+\ell > K+L) \end{cases} = \tilde{M}^{K-k}\mathcal{L}_2$$

holds in the sense of operator for t^i. This is Lemma 2 (iii).

In the case (iv), $k > K$ and $\ell > L$ hold. Then we have

$$\begin{aligned}
&\partial_t^{k-K}\partial_x^{\ell-L}W(t,x)\\
&\quad = \partial_t^{k-K}\partial_x^{\ell-L}\sum_{i\geq K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{W_{iJ}}{(R-x)^{J+K+L+1}}t^i\\
&\quad = \sum_{i\geq K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{C_2 W_{iJ}}{(R-x)^{J+K+L+1+(\ell-L)}}t^{i+K-k}\\
&\quad \ll \sum_{i\geq K_0}\sum_{J=i}^{Mi-(M-1)K_0}\frac{i^{k-K}(J+\ell)^{\ell-L}W_{iJ}}{(R-x)^{J+K+L+1+(\ell-L)}}t^{i+K-k},
\end{aligned}$$

because

$$C_2 := \prod_{q=1}^{k-K}(i+K-k+q)\prod_{q=1}^{\ell-L}(J+L+q) \leq i^{k-K}(J+\ell)^{\ell-L}.$$

Here, since $i \leq Mi \leq Mi + L_1$ and $J + \ell \leq Mi + L_1$, the majorant relation

$$i^{k-K}(J+\ell)^{\ell-L} \ll (Mt\partial_t + L_1)^{k+\ell-K-L} = \mathcal{L}_1$$

holds in the sense of operator for t^i. This is Lemma 2 (iv). □

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