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原著（*Article*）

Gevrey Order of Formal Solutions of Singular First Order Nonlinear Partial Differential Equations of Totally Characteristic Type

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Abstract

This paper is a continuation of the author's previous papers [S2] and [S4].

We consider the following first order nonlinear partial differential equation in general form.

$$f(t, x, u, \partial_t u, \partial_x u) = 0 \quad (t \in \mathbb{C}^d, x \in \mathbb{C}^n).$$

The purpose of this paper is to obtain the estimate of order of divergence for the formal power series solutions of above equation under the condition that the equation is "*singular equation of totally characteristic type*". Such the order of divergence is called "*Gevrey order*", and a characterization result by using the Gevrey order is often called "*Maillet type theorem*". In order to study the Maillet type theorem for above equation, we introduce two matrices. One is introduced by t derivatives, and another is introduced by x derivatives.

In [S2], we studied the convergence of formal solutions in the case where all the eigenvalues of two matrices are not equal to zero and of some conditions. In [S4], we studied the estimate of Gevrey order in the case where the matrix, which is introduced by x derivatives, is nilpotent.

In this paper, we shall study the estimate of Gevrey order in the case where the matrix, which is introduced by t derivatives, is nilpotent.

Key words and Phrases. partial differential equations, complex variables, singular equations, formal solutions, Gevrey order, Maillet type theorem.

1 Main Theorem

Let $\mathbb{C}$ be the set of complex numbers, and $t = (t_1, t_2, \cdots, t_d) \in \mathbb{C}^d$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$. We consider the following first order nonlinear partial differential equation.

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$$(1.1)\qquad \begin{cases} f(t, x, u(t, x), \partial_t u(t, x), \partial_x u(t, x)) = 0, \\ u(0, x) \equiv 0, \end{cases}$$

where $u(t, x)$ denotes an unknown function, $\partial_t u$ and $\partial_x u$ denote

$$\partial_t u = \left(\frac{\partial u}{\partial t_1}, \cdots, \frac{\partial u}{\partial t_d}\right), \quad \partial_x u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$$

respectively. Moreover, for $\tau = (\tau_1, \cdots, \tau_d) \in \mathbb{C}^d$, $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{C}^n$, we assume that the function $f(t, x, u, \tau, \xi)$ is holomorphic in a neighborhood of the origin of $\mathbb{C}^d \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^d \times \mathbb{C}^n$. Especially, we assume that f is an entire function in τ variables for any fixed t, x, u and ξ.

Throughout this paper, we assume the following three assumptions.

Assumption 1 (Singular equations). The function $f(t, x, u, \tau, \xi)$ is singular in t variables in the sense that

$$(1.2)\qquad f(0, x, 0, \tau, 0) \equiv 0 \quad (\forall x \in \mathbb{C}^n \text{ near } x = 0, \ \forall \tau \in \mathbb{C}^d).$$

Assumption 2 (Existence of formal solutions). The equation (1.1) has a formal solution of the form

$$(1.3)\qquad u(t, x) = \sum_{j=1}^{d} \varphi_j(x) t_j + \sum_{|\alpha| \geq 2,\ |\beta| \geq 0} u_{\alpha, \beta} t^\alpha x^\beta,$$

where $|\alpha|$ denotes $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, $(\mathbb{N} = \{0, 1, 2, \cdots\})$, $|\beta|$ denotes $|\beta| = \beta_1 + \cdots + \beta_n$ for $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n$, and $t^\alpha = t_1^{\alpha_1} \cdots t_d^{\alpha_d}$, $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$. Moreover, $\varphi_j(x)$ is holomorphic in a neighborhood of $x = 0$ for all $j = 1, 2, \cdots, d$.

Assumption 3 (Totally characteristic type). The equation (1.1) is of totally characteristic type, that is, $f(t, x, u, \partial_t u, \partial_x u)$ satisfies the following conditions.

$$(1.4)\qquad \begin{cases} f_{\xi_k}(0, x, 0, \{\varphi_j(x)\}, 0) \not\equiv 0 \\ f_{\xi_k}(0, 0, 0, \{\varphi_j(0)\}, 0) = 0 \end{cases} \quad \text{for } k = 1, 2, \cdots, n.$$

Remark 1. By the existence of formal solution, $\{\varphi_j(x)\}$ satisfy the following system of partial differential equations under the assumptions 1, 2 and 3.

$$(1.5)\qquad \begin{aligned} &\frac{\partial}{\partial t_i} f(t, x, u(t, x), \partial_t u(t, x), \partial_x u(t, x))\Big|_{t=0} \\ &= f_{t_i}(0, x, 0, \{\varphi_j(x)\}, 0) + f_u(0, x, 0, \{\varphi_j(x)\}, 0)\varphi_i(x) \\ &+ \sum_{k=1}^{n} f_{\xi_k}(0, x, 0, \{\varphi_j(x)\}, 0) \frac{\partial \varphi_i}{\partial x_k}(x) = 0 \quad (i = 1, 2, \cdots, d) \end{aligned}$$

In the case $d = 1$ (d is the dimension of t variables), a sufficient condition for the formal solution of (1.5) to converge obtained by Miyake and Shirai ([MS1]). In the case $d \geq 2$, a sufficient condition obtained by Shirai ([S2]).

Now we put $\boldsymbol{a}(x)=(0, x, 0, \{\varphi_j(x)\}, 0)$ for the sake of simplicity of notation, and we define functions $a_{i,j}(x)$ $(i, j=1, \cdots, d)$ by

$$(1.6)\qquad a_{i,j}(x):=f_{t_i,\tau_j}(\boldsymbol{a}(x))+f_{u,\tau_j}(\boldsymbol{a}(x))\varphi_i(x)+\sum_{k=1}^{n} f_{\tau_j,\xi_k}(\boldsymbol{a}(x))\frac{\partial\varphi_i}{\partial x_k}(x).$$

Moreover, we define $b_k(x)$ $(k=1, \cdots, n)$ by

$$(1.7)\qquad b_k(x):=f_{\xi_k}(\boldsymbol{a}(x)).$$

Remark 2. By the assumptions, the functions $a_{i,j}(x)$ and $b_k(x)$ are holomophic in a neighborhood of $x=0$, and $b_k(x)$ satisfies $b_k(x)\not\equiv 0$, $b_k(0)=0$ for all $k=1, 2, \cdots, n$.

Our main theorem in this paper is stated as follows.

Theorem 1 (Main Theorem). *Suppose the Assumptions 1, 2 and 3. We assume that all the eigenvalues of* $(a_{i,j}(0))_{i,j=1,\cdots,d}$ *are equal to zero, that is, the Jordan canonical form of* $(a_{i,j}(0))_{i,j=1,\cdots,d}$ *is written by* $\mathrm{diag}(N_1, \cdots, N_I)$ *where* N_j $(j=1, \cdots, I)$ *denotes the following Jordan block of size* $d_j\in\mathbb{N}_+=\{1, 2, 3, \cdots\}$.

$$N_j=\left.\begin{pmatrix} 0 & & & \\ \delta & 0 & & \\ & \ddots & \ddots & \\ & & \delta & 0 \end{pmatrix}\right\} d_j$$

Moreover, we assume that the eigenvalues $\{\mu_k\}_{k=1,\cdots,n}$ *of the Jacobi matrix* $\dfrac{\partial(b_1, \cdots, b_n)}{\partial(x_1, \cdots, x_n)}(0)$ *satisfy the following nonresonance-Poincaré condition*

$$(1.8)\qquad \left|\sum_{k=1}^{n}\mu_k\beta_k+f_u(\boldsymbol{a}(0))\right|\geq C(|\beta|+1)$$

by some positive constant $C>0$ *independent of* $\beta\in\mathbb{N}^n$.

Let $d_0=\max\{d_1, \cdots, d_I\}(\geq 1)$. *Then the formal solution* $u(t, x)$ *belongs to the Gevrey class of order at most* $(2d_0, d_0+1)$, *that is, for the formal solution* $u(t, x)=\sum_{\alpha\in\mathbb{N}^d, \beta\in\mathbb{N}^n}u_{\alpha,\beta}t^\alpha x^\beta$, *the power series*

$$\sum_{\alpha\in\mathbb{N}^d, \beta\in\mathbb{N}^n}\frac{u_{\alpha,\beta}}{|\alpha|!^{2d_0-1}|\beta|!^{d_0}}t^\alpha x^\beta$$

is convergent in a neighborhood of the origin.

2　Related Results

For the formal solution $u(t, x)$, we put $v(t, x)=u(t, x)-\sum_{j=1}^{d}\varphi_j(x)t_j=O(|t|^K)$ $(K\geq 2)$ as a new unknown function. By substituting $u=v+\sum_{j=1}^{d}\varphi_j(x)t_j$ into the equation (1.1), $v(t,x)$ satisfies the following nonlinear partial differential equation.

$$(2.1)\qquad \begin{cases} \left(\sum_{i,j=1}^{d} a_{i,j}(x)t_i\partial_{t_j}+\sum_{k=1}^{n} b_k(x)\partial_{x_k}+f_u(\boldsymbol{a}(x))\right)v(t,x) \\ \quad =\sum_{|\alpha|=K} d_\alpha(x)t^\alpha+f_{K+1}(t,x,v(t,x),\partial_t v(t,x),\partial_x v(t,x)), \\ v(t,x)=O(|t|^K), \end{cases}$$

where $d_\alpha(x)$ denotes a holomorphic function in a neighborhood of the origin, and $f_{K+1}(t, x, v, \tau, \xi)$ is also holomorphic in a neighborhood of the origin with the following Taylor expansion.

$$(2.2)\qquad f_{K+1}(t,x,v,\tau,\xi)=\sum_{V(\alpha,p,q,r)\geq K+1} f_{\alpha pqr}(x)t^\alpha v^p\tau^q\xi^r.$$

Here we used the following notation.

$$(2.3)\qquad V(\alpha,p,q,r)=|\alpha|+Kp+(K-1)|q|+K|r|,$$

which denotes the vanishing order in t variables for each terms $f_{\alpha pqr}(x)t^\alpha v^p\tau^q\xi^r$.

For the equation (2.1), if $b_k(x)\equiv 0\,(k=1,2,\cdots,n)$, the equation is called "*singular equations in t*" or "*partial diffierential equations with degenerate vector field in t*". In this case, many mathematicians obtained a lot of Maillet type theorems. For example, in the papers [GT], [MS1] and [MS2], the various kinds of above cases were studied.

In the case of $b_k(x)\not\equiv 0$ and $b_k(0)=0$, which is the definition of totally characteristic type, Chen and Tahara studied the convergence of formal solution in the case where $(t,x)\in\mathbb{C}^2$ and $b_k(x)=O(x)$ ([CT]). This result was extended to the case of several space variables by Chen and Luo ([CL]). Moreover, these results were extended by Shirai to the case of several time-space variables ([S2]).

On the other hand, Chen, Luo and Tahara studied the Maillet type theorem in the case of $(t,x)\in\mathbb{C}^2$ and $b_k(x)=O(x^K)\,(K\geq 2)$ ([CLT]). Their Maillet type theorem was extended by Shirai to the case of several time-space variables ([S4]).

The statements of [S2] and [S4] are written as follows.

- In [S2], we assume that all eigenvalues $\{\lambda_j\}_{j=1,2,\cdots,d}$ of $(a_{i,j}(0))_{i,j=1,2,\cdots,d}$ and all eigenvalues $\{\mu_k\}_{k=1,\cdots,n}$ of the Jacobi matrix $\dfrac{\partial(b_1,\cdots,b_n)}{\partial(x_1,\cdots,x_n)}(0)$ are not equal to zero, and they satisfy the Poincaré condition $\mathrm{Ch}(\{\lambda_j\},\{\mu_k\})\not\ni 0$, where Ch $\{\cdots\}$ denotes the convex hull of $\{\cdots\}$. Then the formal solution converges in a neighborhood of the origin.
- In [S4], if all eigenvalues $\{\lambda_j\}_{j=1,\cdots,d}$ of $(a_{i,j}(0))_{i,j=1,2,\cdots,d}$ satisfy the Poincaré condi-

tion $\mathrm{Ch}(\{\lambda_j\}) \not\ni 0$, and all eigenvalues $\{\mu_k\}_{k=1,\cdots,n}$ of the Jacobi matrix $\dfrac{\partial(b_1,\cdots,b_n)}{\partial(x_1,\cdots,x_n)}(0)$ are equal to zero, then the formal solution belongs to the Gevrey class of order at most $2\times d_0$ in (t,x), where d_0 denotes the maximum of size of nilpotent Jordan blocks of $\dfrac{\partial(b_1,\cdots,b_n)}{\partial(x_1,\cdots,x_n)}(0)$.

3　Refinement of Theorem 1

In order to prove Theorem 1, we would like to estimate the Gevrey order in each variables $(t_1,\cdots,t_d,x_1,\cdots,x_n)$ of formal solution of (2.1). To do so, we rewrite (2.1).

First, we divide $a_{i,j}(x)$ into the constant part $a_{i,j}(0)$ and vanishing part $\hat{a}_{i,j}(x)$, that is, we set $\hat{a}_{i,j}(x)=a_{i,j}(0)-a_{i,j}(x)=O(|x|)$. Then the vector field with respect to t variables is written by

$$\sum_{i,j=1}^{d} a_{i,j}(x)t_i\partial_{t_j}=(t_1,\cdots,t_d)\begin{pmatrix} a_{1,1}(0) & \cdots & a_{1,d}(0)\\ \vdots & \ddots & \vdots\\ a_{d,1}(0) & \cdots & a_{d,d}(0)\end{pmatrix}\begin{pmatrix}\partial_{t_1}\\ \vdots\\ \partial_{t_d}\end{pmatrix}-\sum_{i,j=1}^{d}\hat{a}_{i,j}(x)t_i\partial_{t_j}.$$

Here we introduce new variables $\tau=(\tau^{(1)},\cdots,\tau^{(I)})\in\mathbb{C}^d$, $(\tau^{(j)}=(\tau_{j,1},\cdots,\tau_{j,d_j})\in\mathbb{C}^{d_j}$, $d=d_1+\cdots+d_I$) by

$$(\tau^{(1)},\cdots,\tau^{(I)})=(t_1,\cdots,t_d)P,$$

where P denotes a regular matrix, which brings $(a_{i,j}(0))$ to the Jordan canonical form. By this linear change of variables, the vector field is transformed as follows.

$$\sum_{i,j=1}^{d} a_{i,j}(x)t_i\partial_{t_j}\mapsto(\tau^{(1)},\cdots,\tau^{(I)})\begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_I\end{pmatrix}\begin{pmatrix}\partial_{\tau^{(1)}}\\ \vdots\\ \partial_{\tau^{(I)}}\end{pmatrix}-\sum_{i,j,k,l}\alpha_{ijkl}(x)\tau_{i,j}\partial_{\tau_{k,l}},$$

where $\alpha_{ijkl}(x)=O(|x|)$ denote holomoprhic functions.

Next, we divide the differential operator with respect to x variables into the following form.

$$\sum_{k=1}^{n} b_k(x)\partial_{x_k}=(x_1,\ \cdots,x_n)\frac{\partial(b_1,\cdots,b_n)}{\partial(x_1,\cdots,x_n)}(0)\begin{pmatrix}\partial_{x_1}\\ \vdots\\ \partial_{x_n}\end{pmatrix}-\sum_{k=1}^{n}\hat{b}_k(x)\partial_{x_k},$$

where $\hat{b}_k(x)$ $(k=1,\ \cdots,\ n)$ denotes a holomorphic function whose vanishing order is greater than or equal to 2. Then we define new variables $\xi=(\xi_1,\cdots,\xi_n)\in\mathbb{C}^n$ by

$$(\xi_1,\cdots,\xi_n)=(x_1,\cdots,x_n)Q,$$

where Q denotes a regular matrix, which brings the Jacobi matrix $\dfrac{\partial(b_1,\cdots,b_n)}{\partial(x_1,\cdots,x_n)}(0)$ to the Jordan canonical form. By this linear change of variables x, the differential operator is written by

$$\sum_{k=1}^{n} b_k(x)\partial_{x_k} \mapsto \sum_{k=1}^{n}\mu_k\xi_k\partial_{\xi_k} + \sum_{k=1}^{n-1}\nu_k\xi_{k+1}\partial_{\xi_k} - \sum_{k=1}^{n}\beta_k(\xi)\partial_{\xi_k},$$

where $\nu_k\,(k=1, \cdots, n-1)$ denotes a nilpotent component of the Jordan canonical form, and $\beta_k(x) = O(|x|^2)\,(k=1, \cdots, n)$ denotes a holomorphic function whose vanishing order is greater than or equal to 2.

Here we rewrite as t instead of τ, and as x instead of ξ again. Then the equation (2.1) is rewritten as follows.

$$\text{(3.1)}\qquad \begin{cases} (\mathcal{N}+\mathcal{D}+\Delta)v = \displaystyle\sum_{i,j,k,l}\alpha_{ijkl}(x)t_{i,j}\partial_{t_{k,l}}v + \sum_{k=1}^{n}\beta_k(x)\partial_{x_k}v \\ \qquad\qquad + \eta(x)v + \displaystyle\sum_{|\alpha|=K}\zeta_\alpha(x)t^\alpha + g_{K+1}(t,x,v,\partial_t v,\partial_x v), \\ v(t,x) = O(|t|^K), \end{cases}$$

where the operators $\mathcal{N}$, $\mathcal{D}$ and Δ denote

$$\text{(3.2)}\qquad \mathcal{N} = \sum_{j=1}^{I}\sum_{k=1}^{d_j-1}\delta t_{j,k+1}\partial_{t_{j,k}},$$

$$\text{(3.3)}\qquad \mathcal{D} = \sum_{k=1}^{n}\mu_k x_k\partial_{x_k} + f_u(\boldsymbol{a}(0)),$$

$$\text{(3.4)}\qquad \Delta = \sum_{k=1}^{n-1}\nu_k x_{k+1}\partial_{x_k}$$

respectively. Moreover, $\eta(x)$ denote $\eta(x) = f_u(\boldsymbol{a}(0)) - f_u(\boldsymbol{a}(x)) = O(|x|)$, and $g_{k+1}(t,x,v,\tau,\xi)$ is holomorphic in a neighborhood of the origin with the same Taylor expansion as (2.2).

In order to state the refinement result, we prepare some notations and deffinitions.

Definition 1 (Borel transform). Let $\mathbb{R}_{\ge 1} = \{x\in\mathbb{R} \mid x\ge 1\}$, *and let* $\boldsymbol{s} = (s_1, \cdots, s_d) \in (\mathbb{R}_{\ge 1})^d$, $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n) \in (\mathbb{R}_{\ge 1})^n$. *For a formal power series* $u(t,x) = \sum u_{\alpha,\beta}t^\alpha x^\beta$, *we define "$\boldsymbol{s}$-Borel transform in t", "σ-Borel transform in x" and "$(\boldsymbol{s},\sigma)$-Borel transform in (t,x)" as follows respectively.*

- *$\boldsymbol{s}$-Borel trans. in t* $\qquad B_t^{\boldsymbol{s}}(u)(t,x) = \sum\dfrac{u_{\alpha,\beta}|\alpha|!}{(\boldsymbol{s}\cdot\alpha)!}t^\alpha x^\beta,$
- *σ-Borel trans. in x* $\qquad B_x^{\sigma}(u)(t,x) = \sum\dfrac{u_{\alpha,\beta}|\beta|!}{(\boldsymbol{\sigma}\cdot\beta)!}t^\alpha x^\beta,$
- *$(\boldsymbol{s},\sigma)$-Borel trans. in (t,x)* $\qquad B_{t,x}^{(\boldsymbol{s},\sigma)}(u)(t,x) = \sum\dfrac{u_{\alpha,\beta}|\alpha|!|\beta|!}{(\boldsymbol{s}\cdot\alpha)!(\boldsymbol{\sigma}\cdot\beta)!}t^\alpha x^\beta,$

where $\boldsymbol{s}\cdot\alpha$ denotes $\boldsymbol{s}\cdot\alpha = s_1\alpha_1 + \cdots + s_d\alpha_d$ ($\boldsymbol{\sigma}\cdot\beta$ is also the same definition), and $a! = \Gamma(a+1)$.

Remark 3. It is trivial that $B_{t,x}^{(\boldsymbol{s},\sigma)}(u) = (B_t^{\boldsymbol{s}}\circ B_x^{\sigma})(u) = (B_x^{\sigma}\circ B_t^{\boldsymbol{s}})(u)$, and $B_t^{\boldsymbol{s}}(u) = B_{t,x}^{(\boldsymbol{s},\mathbf{1}_n)}(u)$, $B_x^{\sigma}(u) = B_{t,x}^{(\mathbf{1}_d,\sigma)}(u)$ for $\mathbf{1}_k = (1, 1, \cdots, 1) \in \mathbb{N}^k$.

Definition 2 (Gevrey class). *We say that a formal power series $u(t,x) = \sum u_{\alpha,\beta}t^\alpha x^\beta$ belongs to the Gevrey class $G_{t,x}^{(\boldsymbol{s},\sigma)}$ of order $(\boldsymbol{s},\sigma)$, if its $(\boldsymbol{s},\sigma)$-Borel transform $B_{t,x}^{(\boldsymbol{s},\sigma)}(u)(t,x)$*

is convergent in a neighborhood of the origin.

Remark 4. (1) By an easy calculation, the following relation holds.

$$B_{t,x}^{(\boldsymbol{s},\sigma)}(u)(t,x) \in G_{t,x}^{(\boldsymbol{s}',\sigma')} \Longrightarrow u(t,x) \in G_{t,x}^{(\boldsymbol{s}+\boldsymbol{s}'-\mathbf{1}_d,\,\sigma+\sigma'-\mathbf{1}_n)}.$$

(2) For $(t,x) \in \mathbb{C}^2$, we consider the following three (formal) power series.

$$\sum \frac{u_{\alpha,\beta}\,\alpha!\beta!}{(s\alpha)!(\sigma\beta)!} t^\alpha x^\beta, \quad \sum \frac{u_{\alpha,\beta}(\alpha+\beta)!}{(s\alpha+\sigma\beta)!} t^\alpha x^\beta, \quad \sum \frac{u_{\alpha,\beta}}{\alpha!^{s-1}\beta!^{\sigma-1}} t^\alpha x^\beta.$$

The representations of these series are different each other, but the properties of convergence or divergence with Gevrey order are equivalent. Indeed, if one is convergent in a neighborhood of the origin, then the others are also convergent in a neighborhood of the origin, and if one is divergent with Gevrey order (s,σ), then the Gevrey orders of the other series are also the same.

By using these notations and defnitions, we obtain the following result.

Theorem 2. *Let* $\boldsymbol{s}^j = (1, 2, \cdots, d_j) \in \mathbb{N}^{d_j} (j = 1, 2, \cdots, I)$*, and let* $d_0 = \max\{d_1, d_2, \cdots, d_I\}$. *Then under the assumptions 1, 2, 3 and the nonresonance-Poincaré condition (1.8), the formal solution of (3.1) belongs to the Gevrey class of order at most* $(\boldsymbol{s}', \sigma')$*, where* $\boldsymbol{s}'$ *and* σ' *denote*

$$\begin{cases} \boldsymbol{s}' = (\boldsymbol{s}^1, \boldsymbol{s}^2, \cdots, \boldsymbol{s}^I) + (d_0, d_0, \cdots, d_0) \in \mathbb{N}^d, \\ \sigma' = (d_0+1, d_0+1, \cdots, d_0+1) \in \mathbb{N}^n. \end{cases} \tag{3.5}$$

Proof of Theorem 1 (Main Theorem). Theorem 1 is proved by Theorem 2 immediately. Indeed, all the components of $(\boldsymbol{s}^1, \cdots, \boldsymbol{s}^I)$ are estimated by d_0. Therefore, all the components of $\boldsymbol{s}'$ are estimated by $2d_0$, which is the consequence of Theorem 1.

4　Sketch of the Proof of Theorem 2

We defne the set of homogeneous polynomials of degree L in t and degree M in x by

$$\mathbb{C}[t]_L[x]_M = \left\{ \sum_{|\alpha|=L,\,|\beta|=M} u_{\alpha,\beta} t^\alpha x^\beta \,\middle|\, u_{\alpha,\beta} \in \mathbb{C} \right\}. \tag{4.1}$$

First we give a following lemma.

Lemma 1. (1) *The operator* $P := \mathcal{N} + \mathcal{D} + \varDelta$ *is invertible on* $\mathbb{C}[t]_L[x]_M$ *for all* $L \geq K$ *and* $M \geq 0$.

(2) *Let* $\boldsymbol{s} = (\boldsymbol{s}^1, \cdots, \boldsymbol{s}^I) \in \mathbb{N}^d (\boldsymbol{s}^j = (1, 2, \cdots, d_j) \in \mathbb{N}^{d_j})$*, and we put* $T = t_1 + \cdots + t_d \in \mathbb{C}$, $X = x_1 + \cdots + x_n \in \mathbb{C}$. *Then, for* $u(t,x) \in \mathbb{C}[t]_L[x]_M$*, if a majorant relation* $B_t^{\boldsymbol{s}}(u)(t,x) \ll W_{L,M} T^L X^M$ $(W_{L,M} \geq 0)$ *holds, then there exists a positive constant* $C_0 > 0$ *independent of L and M such that the following majorant relation holds.*

$$B_t^{s}(P^{-1}u)(t,x) \ll \frac{C_0}{M+1} W_{L,M} T^L X^M \tag{4.2}$$
$$= C_0 (X\partial_X + 1)^{-1} W_{L,M} T^L X^M.$$

By Lemma 1, the operator P is invertible on $\mathbb{C}[[t,x]]_K$ which denotes the set of formal power series whose vanishing order in t is greater than or equal to K as follows.

$$\mathbb{C}[[t,x]]_K = \bigcup_{L\geq K, M\geq 0} \mathbb{C}[t]_L[x]_M.$$

Here we put $U(t,x) = Pv(t,x)$ as a new unknown function. Then $U(t,x)$ satisfies the following.

$$\begin{cases} U(t,x) = \sum\limits_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1}U + \sum\limits_{k=1}^{n} \beta_k(x) \partial_{x_k} P^{-1}U \\ \qquad + \eta(x) P^{-1}U + \sum\limits_{|\alpha|=K} \zeta_\alpha(x) t^\alpha \\ \qquad + g^{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U), \\ U(t,x) = O(|t|^K). \end{cases} \tag{4.3}$$

For the equation (4.3), we apply the Borel transform of order $\boldsymbol{s} = (\boldsymbol{s}^1, \cdots, \boldsymbol{s}^I) \in \mathbb{N}^d (\boldsymbol{s}^j = (1, 2, \cdots, d_j) \in \mathbb{N}^{d_j})$ in t, then (4.3) is reduced to the following.

$$\begin{aligned} B_t^{s}(U)(t,x) &= B_t^{s}\Big(\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1}U\Big) \\ &\quad + B_t^{s}\Big(\sum_{k=1}^{n} \beta_k(x) \partial_{x_k} P^{-1}U\Big) \\ &\quad + B_t^{s}(\eta(x) P^{-1}U) + \sum_{|\alpha|=K} \frac{\zeta_\alpha(x)|\alpha|!}{(\boldsymbol{s}\cdot\alpha)!} t^\alpha \\ &\quad + \mathrm{B}_t^{s}\{g_{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U)\}. \end{aligned} \tag{4.4}$$

In order to estimate the Borel transforms of products and derivatives with respect to t or x by using majorant series, we give the following lemma.

Lemma 2. *Let* $\boldsymbol{s} = (\boldsymbol{s}^1, \cdots, \boldsymbol{s}^I) \in \mathbb{N}^d (\boldsymbol{s}^j = (1, 2, \cdots, d_j) \in \mathbb{N}^{d_j})$.

(1) *For two formal power series* $u(t,x)$, $v(t,x) \in \mathbb{C}[[t,x]]_0$, *there exists a positive constant* $C_1 > 0$ *depends only on s such that the following majorant relation holds.*

$$B_t^{s}(uv)(t,x) \ll C_1 B_t^{s}(|u|)(t,x) \times B_t^{s}(|v|)(t,x). \tag{4.5}$$

Here $|u|(t,x)$ *is defined as follows: for* $u(t,x) = \sum u_{\alpha\beta} t^\alpha x^\beta$, $|u|(t,x)$ *is defined by* $|u|(t,x) = \sum |u_{\alpha\beta}| t^\alpha x^\beta$.

(2) *Let* $T = t_1 + \cdots + t_d$, $X = x_1 + \cdots + x_n$, *and let* $W(T,X)$ *be a formal power series in* T *and* X. *If* $B_t^{s}(u)(t,x) \ll W(T,X)$, *then there exists a positive constant* $C_2 > 0$ *such that the following majorant relations hold.*

$$B_t^{s}(\partial_{t_{i,j}} P^{-1}u)(t,x) \ll C_2 \partial_T (T\partial_T)^{j-1} W(T,X), \tag{4.6}$$

$$B_t^{s}(\partial_{x_k} P^{-1}u)(t,x) \ll C_2 \partial_X (X\partial_X + 1)^{-1} W(T,X) \ll C_2 \times S(W)(T,X), \tag{4.7}$$

where $S(W)(T,X)$ is called the shift function in X of $W(T,X)$. The definition of shift function in X is as follows.

(4.8) $$S(W)(T,X)=\frac{W(T,X)-W(T,0)}{X}.$$

By Lemma 2, if a majorant relation $B_t^s(U)(t,x)\ll W(T,X)$ holds between $B_t^s(U)(t,x)$ and some formal series $W(T,X)$, then there exists a positive constant $C_3>0$ such that the following majorant relations hold.

(4.9) $$B_t^s(\alpha_{ijkl}(x)t_{i,j}\partial_{t_{k,l}}P^{-1}U)(t,x)\ll C_3|\alpha_{ijkl}|(X)(T\partial_T)^l W(T,X),$$

(4.10) $$B_t^s(\beta_k(x)\partial_{x_k}P^{-1}U)(t,x)\ll C_3|\beta_k|(X)S(W)(T,X),$$

(4.11) $$B_t^s(\eta(x)P^{-1}U)(t,x)\ll C_3|\eta|(X)W(T,X),$$

(4.12) $$B_t^s\left(\sum_{|\alpha|=K}\frac{\zeta_\alpha(x)|\alpha|!}{(\boldsymbol{s}\cdot\alpha)!}t_\alpha\right)\ll\left(\sum_{|\alpha|=K}|\zeta_\alpha|(X)\right)T^K=:\zeta(X)T^K,$$

(4.13) $$\begin{aligned}&B_t^s(g_{K+1}(t,x,P^{-1}U,\partial_tP^{-1}U,\partial_xP^{-1}U))\\&\quad\ll|g_{K+1}|(T,X,C_3W,\{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j},\{C_3S(W)\}_k).\end{aligned}$$

We remark that j, which appeared in the right hand side of (4.13), is less than or equal to $\max\{d_1,\cdots,d_l\}$. Moreover, since $W(T,X)$ is a majorant series of $B_t^s(P^{-1}u)(t,x)$, then we have $W(T,X)\gg 0$. Therefore, by using this property, we obtain the following majorant relation.

(4.14) $$XS(W)(T,X)=W(T,X)-W(T,0)\ll W(T,X).$$

For $|\beta_k|(X)=O(X^2)$, we put a holomorphic function $|\hat{\beta}_k|(X)$ by $|\beta_k|(X)/X=O(X)$. Then a majorant relation

$$|\beta_k|(X)S(W)=\frac{|\beta_k|(X)}{X}\cdot XS(W)\ll\frac{|\beta_k|(X)}{X}W=:|\hat{\beta}_k|(X)W$$

holds.

We consider the following equation.

(4.15) $$\begin{aligned}W=&\sum_{i,j,k,l}\tilde{\alpha}_{ijkl}(X)(T\partial_T)^lW+\sum_{k=1}^n\tilde{\beta}_k(X)W\\&+\tilde{\eta}(X)W+\zeta(X)T^K\\&+|g_{K+1}|(T,X,C_3W,\{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j},\{C_3S(W)\}_k),\end{aligned}$$

where $\tilde{\alpha}_{ijkl}(X)=C_3|\alpha_{ijkl}|(X)$, $\tilde{\beta}_k(X)=C_3|\hat{\beta}_k|(X)$, $\tilde{\eta}(X)=C_3|\eta|(X)$. These are all holomorphic functions in a neighborhood of $X=0$ and vanish at $X=0$. By the construction of this equation, it is easily see that

$$B_t^s(U)(t,x)\ll W(T,X).$$

Here we put $F(X)=1-\sum_{k=1}^n\tilde{\beta}_k(X)-\tilde{\eta}(\mathrm{X})$. Since $F(0)=1\neq 0$, $1/F(X)$ is holomorphic in a neighborhood of $X=0$. Therefore, by multiplying $1/F(X)$ to the both sides, the equation (4.15) is reduced to the following.

$$W = \sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X)(T\partial_T)^l W + \hat{\zeta}_\alpha(X)T^K + G_{K+1}(T, X, C_3 W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3 S(W)\}k), \tag{4.16}$$

where $\hat{\alpha}_{ijkl}(X) = \tilde{\alpha}_{ijkl}(X)/F(X) = O(X)$, $\hat{\beta}_k(X) = \tilde{\beta}_k(X)/F(X) = O(X)$, $\hat{\eta}(X) = \tilde{\eta}(X)/F(X) = O(X)$, $G_{K+1}(T, X, u, \tau, \xi) = |g_{K+1}|(T, X, u, \tau, \xi)/F(\mathrm{X})$.

For the equation (4.16), the following lemma holds.

Lemma 3. *The formal solution $W(T, X)$ of (4.16) belongs to the Gevrey class $G_{T,X}^{(d_0+1, d_0+1)}$.*

Remark 5. In this paper, we omit the detail of proof of Lemma 3, but in the next section, we shall treat the proof of Lemma 3 in the example case which is a typical equation for (4.16).

By Lemma 3, $W(T, X) \in G_{T,X}^{(d_0+1, d_0+1)}$. On the other hand, for $\boldsymbol{s} = (\boldsymbol{s}^1, \cdots, \boldsymbol{s}^I)$ $(\boldsymbol{s}^j = (1, 2, \cdots, d_j))$, the majorant relation $B_t^{\boldsymbol{s}}(U)(t, x) \ll W(T, X)$ holds. By combining these properties, we have

$$B_t^{\boldsymbol{s}}(U)(t, x) = B_{t,x}^{(\boldsymbol{s};\, \mathbf{1}_n)}(U)(t, x) \in G_{T,X}^{(d_0+1, d_0+1)}.$$

Therefore, by Remark 4, the Gevrey orders $(\boldsymbol{s}', \sigma')$ of $U(t, x)$ is obtained by

$$\boldsymbol{s}' = \boldsymbol{s} + (d_0+1, \cdots, d_0+1) - \mathbf{1}_d = \boldsymbol{s} + (d_0, \cdots, d_0),$$
$$\sigma' = \mathbf{1}_n + (d_0+1, \cdots, d_0+1) - \mathbf{1}_n = (d_0+1, \cdots, d_0+1),$$

which is a desired orders in Theorem 2.

5 The Proof of Lemma 3 in Typical Case

We consider the following nonlinear partial differential equation, which is a simple example of (4.16).

$$\begin{cases} W(T, X) = \hat{\alpha}(X)(T\partial_T)^{d_0} W(T, X) + \hat{\zeta}(X)T^K \\ \qquad\qquad + \partial_T(T\partial_T)^{d_0-1}W(T, X) \times S(W)(T, X) \\ W(T, X) = O(T^K), \end{cases} \tag{5.1}$$

where $\hat{\alpha}(X)$ is holomorphic in a neighborhood of the origin satisfying $\hat{\alpha}(X) = O(X)$.

Lemma 4. *The formal solution $W(T, X)$ of (5.1) belongs to the Gevrey class $G_{T,X}^{(d_0+1, d_0+1)}$, which is the same statement as Lemma 3. Namely, if we write $W(T,X) = \sum_{L \geq K, M \geq 0} W_{L,M} T^L X^M$, then a power series $\sum_{L \geq K, M \geq 0} \frac{W_{L,M}}{(L+M)!^{d_0}} T^L X^M$ converges in a neighborhood of the origin.*

Proof of Lemma 4. We put $W(T, X) = \sum_{L \geq K} W_L(X) T^L$. By substituting this into the equation (5.1), we obtain the following recurrence formulas for the coefficients $\{W_L(X)\}_{L \geq K}$.

The case $L=K$,

(5.2) $$W_K(X)=\hat{\alpha}(X)K^{d_0}W_K(X)+\hat{\zeta}(X),$$

The case $L\geq K+1$,

(5.3) $$W_L(X)=\hat{\alpha}(X)L^{d_0}W_L(X)+\sum_{\substack{(p-1)+q=L\\ p,q\geq K}}p^{d_0}W_p(X)S(W_q)(X).$$

First we consider the equation (5.2). We put the Taylor expansions of $W_K(X)$, $\hat{\alpha}(X)$ and $\hat{\zeta}(X)$ by

$$W_K(X)=\sum_{M\geq 0}W_{K,M}X^M,\quad \hat{\alpha}(X)=\sum_{N\geq 1}\alpha_N X^N,\quad \hat{\zeta}(X)=\sum_{M\geq 0}\zeta_M X^M.$$

Then we have the following recurrence formulas for the coeffcients $\{W_{K,M}\}_{M\geq 0}$.

(5.4) $$W_{K,M}=\sum_{i+j=M,\,i\geq 1}K^{d_0}\alpha_i W_{K,j}+\zeta_M.$$

Here we put $V_{K,M}=W_{K,M}/(K+M)!^{d_0}$, then $\{V_{K,M}\}$ satisfy the following recurrence formulas.

(5.5) $$V_{K,M}=\sum_{i+j=M,\,i\geq 1}\frac{K^{d_0}\alpha_i(K+j)!^{d_0}}{(K+M)!^{d_0}}V_{K,j}+\frac{\zeta_M}{(K+M)!^{d_0}}.$$

By the conditions $i+j=M$ and $i\geq 1$, we have $j\leq M-1$. Therefore, we can estimate the factorial part as follows.

$$\frac{K^{d_0}(K+j)!^{d_0}}{(K+M)!^{d_0}}\leq\left(\frac{K(K+M-1)!}{(K+M)!}\right)^{d_0}\leq 1.$$

By using this estimate, we obtain the following inequality.

$$|V_{K,M}|\leq\sum_{i+j=M,\,i\geq 1}|\alpha_i||V_{K,j}|+|\zeta_M|.$$

Here we consider the following equation.

(5.6) $$Y_K(X)=|\hat{\alpha}|(X)Y_K(X)+|\hat{\zeta}|(X).$$

By the construction of (5.6), we easily see that the following majorant relation holds.

$$Y_K(X)\gg\sum_{M\geq 0}V_{K,M}X^M=\sum_{M\geq 0}\frac{W_{K,M}}{(K+M)!^{d_0}}X^M.$$

Next, we consider (5.3). By substituting $W_L(X)=\sum_{M\geq 0}W_{L,M}X^M$ into (5.3), the coefficients $\{W_{L,M}\}_{M\geq 0}$ satisfy the following recurrence formulas.

(5.7) $$W_{L,M}=\sum_{i+j=M,\,i\geq 1}\alpha_i L^{d_0}W_{L,j}+\sum_{\substack{(p-1)+q=L\\ p,q\geq K}}\sum_{i+(j-1)=M}p^{d_0}W_{p,i}W_{q,j}.$$

We put $V_{L,M}=W_{L,M}/(L+M)!^{d_0}$, then $\{V_{L,M}\}_{M\geq 0}$ satisfy

(5.8) $$V_{L,M}=\sum_{i+j=M,\,i\geq 1}\frac{L^{d_0}(L+j)!^{d_0}}{(L+M)!^{d_0}}\alpha_i V_{L,j}$$
$$+\sum_{\substack{(p-1)+q=L\\ p,q\geq K}}\sum_{i+(j-1)=M}\frac{p^{d_0}(p+i)!^{d_0}(q+j-1)!^{d_0}}{(L+M)!^{d_0}}V_{p,i}V_{q,j}.$$

By the same argument as we considered for (5.2), we can estimate the factorial part in linear term as follows.

$$\frac{L^{d_0}(L+j)!^{d_0}}{(L+M)!^{d_0}} \leq 1.$$

In order to estimate the nonlinear part, we prepare the following lemma.

Lemma 5. *For $n_1, n_2, \cdots, n_p \geq K$ $(n_1, \cdots, n_p \in \mathbb{N})$, the following inequality holds.*

$$n_1!n_2!\cdots n_p! \leq K!^{p-1}(n_1+n_2+\cdots+n_p-K(p-1))! \tag{5.9}$$

Lemma 5 is easily proved by the induction with respect to p. You can find the detail of the proof in [MS1] or [S1].

We remark that the conditions $p \geq K$ and $q+j-1 \geq K$ hold. Then by Lemma 5, we obtain the following inequality.

$$\begin{aligned}\left(\frac{p(p+i)!(q+j-1)!}{(L+M)!}\right)^{d_0} &\leq \left(\frac{p \cdot K!(p+q+i+j-1-K)!}{(L+M)!}\right)^{d_0}\\ &= \left(\frac{p \cdot K!(L+M+1-K)!}{(L+M)!}\right)^{d_0}\\ &\leq K!^{d_0}\left(\frac{L(L+M+1-K)!}{(L+M)!}\right)^{d_0} \leq K!^{d_0} = \text{Constant}.\end{aligned}$$

Therefore, the following estimate holds.

$$|V_{L,M}| \leq \sum_{i+j=M,\, i\geq 1} |\alpha_i||V_{L,j}| + K!^{d_0} \sum_{\substack{(p-1)+q=L\\ p,q\geq K}} \sum_{i+(j-1)=M} |V_{p,i}||V_{q,j}|.$$

We consider the following equation.

$$Y_L(X) = |\hat{\alpha}|(X)Y_L(X) + K!^{d_0} \sum_{\substack{(p-1)+q=L\\ p,q\geq K}} Y_p(X)S(Y_q)(X). \tag{5.10}$$

By using the above estimate, the following majorant relation holds for all $L \geq K+1$.

$$Y_L(X) \gg \sum_{M\geq 0} V_{L,M}X^M = \sum_{M\geq 0} \frac{W_{L,M}}{(L+M)!^{d_0}} X^M.$$

Next, we consider the equation

$$\begin{cases} Y(T,X) = |\hat{\alpha}|(X)Y(T,X) + |\hat{\zeta}|(X)T^K \\ \qquad\qquad + \dfrac{K!^{d_0}}{T} Y(T,X) \times S(Y)(T,X), \\ Y(T,X) = O(T^K). \end{cases} \tag{5.11}$$

We set $Y(T,X) = \sum_{L\geq K} Y_L(X)T^L$ as a formal solution of (5.11). By substituting this into (5.11), we can easily see that the coeffcients $\{Y_L(X)\}$ satisfy the recurrence formulas (5.6) and (5.10). Therefore, $Y(T,X)$ satisfies the following majorant relations.

$$Y(T,X) = \sum_{L\geq K} Y_L(X)T^L \gg \sum_{L\geq K}\sum_{M\geq 0} \frac{W_{L,M}}{(L+M)!^{d_0}} X^M T^L \sim B_{T,X}^{(d_0+1,\, d_0+1)}(W)(T,X).$$

Finally, we shall prove the convergence of $Y(T,X)$. We define a new unknown function $Z(T,X)$ by $Z(T,X) = Y(T,X)/T\,(=O(T^{K-1}))$. Then $Z(T,X)$ satisfies the following equation.

$$(5.12)\qquad \begin{cases} Z(T,X) = |\hat{\alpha}|(X)Z(T,X) + |\hat{\zeta}|(X)T^{K-1} \\ \qquad\qquad + K!^{d_0}Z(T,X) \times TS(Z)(T,X), \\ Z(T,X) = O(T^{K-1}). \end{cases}$$

Since $|\hat{\alpha}|(X)$ satisfies $|\hat{\alpha}|(X) = O(X)$, a function $F(X) = 1/(1-|\hat{\alpha}|(X))$ is holomorphic in a neighborhood of $X = 0$.

By multiplying the holomorphic function $F(X)$ to the both sides, (5.12) is rewritten as follows.

$$(5.13)\qquad Z(T,X) = F(X)|\hat{\zeta}|(X)T^{K-1} + K!^{d_0}F(X)Z(T,X) \times TS(Z)(T,X).$$

In order to prove the convergence of $Y(T, X)$, it is sufficient to prove the convergence of $Z(T,X)$. To do so, we prove that $Z(\rho, \rho)$ $(\rho \in \mathbb{C})$ is convergent in a neighborhood of $\rho = 0$.

In the equation (5.13), we put $T = X = \rho$, then (5.13) is rewritten as follows.

$$(5.14)\qquad Z(\rho,\rho) = F(\rho)|\hat{\zeta}|(\rho)\rho^{K-1} + K!^{d_0}F(\rho)Z(\rho,\rho) \times \rho S(Z)(\rho,\rho).$$

By the majorant relation (4.14), $\rho S(Z)(\rho, \rho) \ll Z(\rho, \rho)$ does holds. Therefore, the formal solution $P(\rho)$ of the following equation is a majorant series of $Z(\rho, \rho)$.

$$(5.15)\qquad P(\rho) = F(\rho)|\hat{\zeta}|(\rho)\rho^{K-1} + K!^{d_0}F(\rho)P(\rho)^2,$$

with $P(\rho) = O(\rho^{K-1})$.

Now, we put

$$H(\rho, P) = P - F(\rho)|\hat{\zeta}|(\rho)\rho^{K-1} - K!^{d_0}F(\rho)P^2.$$

By an easy calculation, $H(\rho, P)$ satisfies the relation

$$H(0,0) = 0 \ \text{ and } \ \frac{\partial H}{\partial P}(0,0) = 1 \neq 0.$$

By the above conditions, we obtain the uniquely existence of holomorphic solution $P(\rho)$ of the equation $H(\rho, P) = 0$ by using the implicit function theorem.

Hence Lemma 4 is proved.

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