原著（*Article*）

Maillet Type Theorem for Nonlinear *q*-Difference-Differential Equations of Kowalevski Type

コワレフスキー型非線形 *q* 差分微分方程式に対するマイエ型定理

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Abstract

The following equations are called the Kowalevski type equations.

$$\begin{cases} \partial_t^m u(t,x) = f\left(t, x, \left\{\partial_x^i \partial_t^{m-j} u(t,x)\right\}_{i \le j,\, 1 \le j \le m}\right), \\ \partial_t^r u(0,x) = \varphi_r(x) \in \mathcal{O}_x \quad (0 \le r \le m-1). \end{cases} \tag{K}$$

In 1870's, S. Kowalevskaya proved the unique solvability of above analytic Cauchy problem (K) on $\mathcal{O}_{x,t}$. This result is now known as Cauchy-Kowalevski's Theorem.

In this paper, we consider the following equations obtained by substituting the differential operators ∂_t^m, ∂_x^i and ∂_t^{m-j} of (K) into $\partial_t^L D_{t,q_2}^K$, $\partial_x^{\ell_1} D_{x,q_1}^{k_1}$ and $\partial_t^{\ell_2} D_{t,q_2}^{k_2}$, respectively.

$$\begin{cases} \partial_t^L D_{t,q_2}^K u(t,x) = f\left(t, x, \left\{\partial_x^{\ell_1} D_{x,q_1}^{k_1} \partial_t^{\ell_2} D_{t,q_2}^{k_2} u(t,x)\right\}_{\Delta}\right), \\ u(t,x) = O(t^{L+K}), \end{cases} \tag{E}$$

where $\Delta = \{(\ell_1, k_1, \ell_2, k_2)\,;\, \ell_1 + k_1 + \ell_2 + k_2 \le L + K,\ \ell_2 + k_2 \le L + K - 1\}$. We call (E) "$q$-difference-differential equations of Kowalevski type".

The purpose of this paper is to give the Maillet type theorem for (E).

Keywords. q-difference-differential equations, Maillet type theorem

1. Introduction

Let $(t,x) \in \mathbb{C}^2$. For $q > 0$ and $q \ne 1$, we define q-difference operators $D_{x,q}$ and $D_{t,q}$ by

$$D_{x,q} u(t,x) = \frac{u(t,x) - u(t,qx)}{(1-q)x} \quad \text{and} \quad D_{t,q} u(t,x) = \frac{u(t,x) - u(qt,x)}{(1-q)t}, \tag{1.1}$$

respectively. Especially, for x^n and t^n $(n = 0, 1, 2, \dots)$, we have

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$$D_{x,q}x^n = \begin{cases} \dfrac{1-q^n}{1-q}x^{n-1} & (n \geq 1), \\ 0 & (n=0) \end{cases} \quad \text{and} \quad D_{t,q}t^n = \begin{cases} \dfrac{1-q^n}{1-q}t^{n-1} & (n \geq 1), \\ 0 & (n=0), \end{cases}$$

respactively. For the sake of simplicity of notation, we put $[n]_q = (1-q^n)/(1-q)$ $(n = 0, 1, 2, \dots)$. By the above definition of q-difference operator in t, we define the inverse q-difference operator $D_{t,q}^{-1}$ by

$$D_{t,q}^{-1}t^n = \frac{1-q}{1-q^{n+1}}t^{n+1} + C(x) = \frac{t^{n+1}}{[n+1]_q} + C(x),$$

where $C(x)$ is a function which corresponds to the integral constant.

If $U(t,x) = D_{t,q}u(t,x)$ and $u(t,x) = \sum_{n\geq 1} u_n(x)t^n = O(t)$, then we have $u(t,x) = D_{t,q}^{-1}U(t,x)$ and, the integral constant $C(x)$ does not appear.

Moreover, if $q \uparrow 1$, then we have

$$\begin{aligned} &D_{x,q}u(t,x) \to \partial_x u(t,x), \\ &D_{t,q}u(t,x) \to \partial_t u(t,x), \quad D_{t,q}^{-1}u(t,x) \to \partial_t^{-1}u(t,x) = \int_0^t u(t,x)dt + C(x). \end{aligned}$$

Throughout this paper, we assume that $0 < q < 1$. In this case, from the inequalities for all $n \in \mathbb{N}_0$

$$[n]_q \leq \frac{1}{1-q} \quad \text{and} \quad \frac{1}{[n]_q} = \frac{1}{1+q+\cdots+q^{n-1}} \leq 1 \leq \frac{1}{1-q},$$

we have the following majorant relation.

$$D_{t,q}^k t^n \ll \left(\frac{1}{1-q}\right)^{|k|} t^{n-k} \quad (n \in \mathbb{N}_0, k \in \mathbb{Z}). \tag{1.2}$$

Here for $u^i(t) = \sum_{n\geq 0} u_n^i t^n$ $(i = 1, 2)$, we define $u^1(t) \ll u^2(t)$ if $|u_n^1| \leq |u_n^2|$ $({}^\forall n \in \mathbb{N}_0)$. For $u^i(t,x) = \sum_{n\geq 0} u_n^i(x)t^n$ $(i = 1, 2)$, we define $u^1(t,x) \ll u^2(t,x)$ if $u_n^1(x) \ll u_n^2(x)$ $({}^\forall n \in \mathbb{N}_0)$.

Let $0 < q_1 < 1$ and $0 < q_2 < 1$. We consider the following nonlinear q-difference-differential equations

$$\begin{cases} \mathcal{D}_{t,q_2}^{L,K}u(t,x) = f\left(t, x, \left\{\mathcal{D}_{x,q_1}^{\ell_1,k_1}\mathcal{D}_{t,q_2}^{\ell_2,k_2}u(t,x)\right\}_\Delta\right), \\ u(t,x) = O(t^{L+K}), \end{cases} \tag{E}$$

where $(t,x) \in \mathbb{C}^2$, $L, K, \ell_1, \ell_2, k_1, k_2 \in \mathbb{N}_0$, $\mathcal{D}_{t,q_2}^{\alpha,\beta} = \partial_t^\alpha D_{t,q_2}^\beta$, $\mathcal{D}_{x,q_1}^{\delta,\gamma} = \partial_x^\delta D_{x,q_2}^\gamma$ and

$$\Delta = \{(\ell_1, k_1, \ell_2, k_2)\,;\, \ell_1 + k_1 + \ell_2 + k_2 \leq L + K, \ell_2 + k_2 \leq L + K - 1\}.$$

Moreover, $f(t, x, \{\xi_{\ell_1k_1\ell_2k_2}\}_\Delta)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$f(t, x, \{\xi_{\ell_1k_1\ell_2k_2}\}_\Delta) = \sum_{p+|\beta|\geq 0} f_{p\beta}(x)t^p \prod_\Delta (\xi_{\ell_1k_1\ell_2k_2})^{\beta_{\ell_1k_1\ell_2k_2}}.$$

Definition 1 (Kowalevski type)
We say that (E) are "q-difference-differential equations of Kowalevski type", if

$$\Delta = \Delta_1 := \{(\ell_1, k_1, \ell_2, k_2)\,;\, 0 \leq \ell_1 + \ell_2 + k_1 + k_2 \leq L + K, 0 \leq \ell_2 + k_2 \leq L + K - 1\}.$$

Especially, we say that (E) are "partial differential equations of Kowalevski type", if

$$\Delta = \Delta_2 := \Delta_1 \cap \{(\ell_1, k_1, \ell_2, k_2)\,;\, 0 \le \ell_1 + \ell_2 \le L, 0 \le \ell_2 \le L-1\}.$$

The following theorems hold.

Theorem 1 *Assume that the equations* (E) *are "partial differential equations of Kowalevski type". Then the formal power series solution* $u = \sum_{n \ge L+K} u_n(x)t^n \in \mathcal{O}_x[[t]]$ *exists uniquely and it is convergent in a neighborhood of the origin.*

Theorem 2 *Assume that the equations* (E) *are "q-difference-differential equations of Kowalevski type", not "partial differential equations of Kowalevski type". Then the formal solution* $u = \sum_{n \ge L+K} u_n(x)t^n \in \mathcal{O}_x[[t]]$ *exists uniquely and it belongs to the Gevrey class* $\mathcal{G}_t^{s+1}$ *of order at most* $s+1$*, where*

$$s = \max_{p+|\beta| \ge 0} \left\{ \frac{M(p,\beta) - L}{V(p,\beta)}, 0 \right\}. \tag{1.3}$$

Here $M(p,\beta)$ *and* $V(p,\beta)$ *are defined as follows. For each term*

$$f_{p\beta}(x)t^p \prod_{\Delta} \left\{ \mathcal{D}_{x,q_1}^{\ell_1,k_1} \mathcal{D}_{t,q_2}^{\ell_2,k_2} u(t,x) \right\}^{\beta_{\ell_1 k_1 \ell_2 k_2}}$$

of Taylor expansion of the righthand side of (E), we define

$$\begin{aligned} M(p,\beta) &= \max\{\ell_1 + \ell_2\,;\, \beta_{\ell_1 k_1 \ell_2 k_2} \ne 0\}, \\ V(p,\beta) &= p + \sum_{\Delta} (L + K - \ell_2 - k_2)\beta_{\ell_1 k_1 \ell_2 k_2} \ (\ge 1). \end{aligned}$$

Moreover, the definition of the Gevrey class of order $s+1$ *is as follows.*

$$u(t,x) = \sum_{n \ge 0} u_n(x)t^n \in \mathcal{G}_t^{s+1} \quad \textit{if} \ \sum_{n \ge 0} \frac{u_n(x)}{n!^s} t^n \in \mathcal{O}_x\{t\}.$$

Example 1 Let $0 < q_1 < 1$ and $0 < q_2 < 1$. We consider the following equation.

$$\begin{cases} \mathcal{D}_{t,q_2}^{4,2} u(t,x) = \dfrac{1}{R-x} + \left(\mathcal{D}_{x,q_1}^{1,1} \mathcal{D}_{t,q_2}^{3,1} u(t,x)\right) \left(\mathcal{D}_{x,q_1}^{2,2} \mathcal{D}_{t,q_2}^{1,1} u(t,x)\right)^2, \\ u(t,x) = O(t^6). \end{cases}$$

This equation is a "partial differential equation of Kowalevski type". Therefore, by Theorem 1, the formal solution is convergent in a neighborhood of the origin.

Example 2 Let $0 < q_1 < 1$ and $0 < q_2 < 1$. We consider the following equation.

$$\begin{cases} \mathcal{D}_{t,q_2}^{2,4} u(t,x) = \dfrac{1}{R-x} + \left(\mathcal{D}_{x,q_1}^{1,1} \mathcal{D}_{t,q_2}^{3,1} u(t,x)\right) \left(\mathcal{D}_{x,q_1}^{2,2} \mathcal{D}_{t,q_2}^{1,1} u(t,x)\right)^2, \\ u(t,x) = O(t^6). \end{cases}$$

This is a "q-difference-differential equation of Kowalevski type", not a "differential equation of Kowalevski type ". Therefore, by Theorem 2, the formal solution belongs to the Gevrey class of order at most

$$s + 1 = \frac{\max\{1+3, 2+1\} - 2}{1 \cdot (2+4-3-1) + 2 \cdot (2+4-1-1)} + 1 = \frac{6}{5}.$$

2. Proof of Theorem 1

Let $U(t,x) = \mathcal{D}_{t,q_2}^{L,K} u(t,x)$ as a new unknown function. Since $u = O(t^{L+K})$, we obtain $u(t,x) = (\mathcal{D}_{t,q_2}^{L,K})^{-1} U(t,x) = D_{t,q_2}^{-K} \partial_t^{-L} U(t,x)$. Then (E) is written as follows.

$$U(t,x) = f\left(t, x, \left\{\mathcal{D}_{x,q_1}^{\ell_1,k_1} \partial_t^{\ell_2} D_{t,q_2}^{k_2-K} \partial_t^{-L} U(t,x)\right\}_{\Delta_2}\right). \tag{E$_1$}$$

For the sake of simplicity of notation, we put $\Lambda_{t,q_2}^{\ell_2,k_2} = \partial_t^{\ell_2} D_{t,q_2}^{k_2-K} \partial_t^{-L}$. In this case, (E$_1$) is rewritten by

$$U(t,x) = f\left(t, x, \left\{\mathcal{D}_{x,q_1}^{\ell_1,k_1} \Lambda_{t,q_2}^{\ell_2,k_2} U(t,x)\right\}_{\Delta_2}\right). \tag{E$_1'$}$$

If $U(t,x) = \sum_{i=0}^{\infty} U_i(x) t^i \ll V(t,x) = \sum_{i=0}^{\infty} V_i(x) t^i$, then, by using (1.2), we obtain the following majorant relation for $\Lambda_{t,q_2}^{\ell_2,k_2} U$.

$$\begin{aligned}
&\Lambda_{t,q_2}^{\ell_2,k_2} U(t,x) = \partial_t^{\ell_2} D_{t,q_2}^{k_2-K} \partial_t^{-L} \sum_{i=0}^{\infty} U_i(x) t^i \\
&\ll \partial_t^{\ell_2} D_{t,q_2}^{k_2-K} \sum_{i=0}^{\infty} V_i(x) \frac{t^{i+L}}{(i+1)\cdots(i+L)} \\
&\ll \partial_t^{\ell_2} \sum_{i=0}^{\infty} \left(\frac{1}{1-q_2}\right)^{|k_2-K|} V_i(x) \frac{t^{i+L-k_2+K}}{(i+1)\cdots(i+L)} \\
&= \left(\frac{1}{1-q_2}\right)^{|k_2-K|} \sum_{i=0}^{\infty} V_i(x) \frac{\prod_{r=1}^{\ell_2}(i+L-k_2+K-r+1)}{\prod_{r=1}^{L}(i+r)} t^{i+L-k_2+K-\ell_2}.
\end{aligned} \tag{2.1}$$

We remark that when (E$_1$) is a "q-difference-differential equation of Kowalevski type", we have $i+L-k_2+K-\ell_2 \geq 1$. For $r = 1, \ldots, \ell_2$,

$$\begin{aligned}
\frac{i+L-k_2+K-r+1}{i+r} &= \frac{(i+r)+(L+K-k_2-2r+1)}{i+r} \\
&= 1 + \frac{L+K-2r+1}{i+r} \\
&\leq 1 + (L+K-1) = L+K.
\end{aligned}$$

Moreover, when (E$_1$) is a "partial differential equation of Kowalevski type", we have $\ell_2 < L$. Then we obtain the following inequality.

$$\begin{aligned}
&\frac{\prod_{r=1}^{\ell_2}(i+L-k_2+K-r+1)}{\prod_{r=1}^{L}(i+r)} \leq \prod_{r=1}^{\ell_2} \frac{i+L-k_2+K-r+1}{i+r} \prod_{r=\ell_2+1}^{L} \frac{1}{i+r} \\
&\leq \prod_{r=1}^{\ell_2}(L+K) \prod_{r=\ell_2+1}^{L} \frac{1}{i+r} \leq (L+K)^L \prod_{r=1}^{L-\ell_2} \frac{1}{i+\ell_2+r} \qquad (\because \ell_2 < L).
\end{aligned}$$

Therefore,

$$\Lambda_{t,q_2}^{\ell_2,k_2} U(t,x) \ll C_{k_2} \sum_{i=0}^{\infty} V_i(x) \frac{t^{i+L-k_2+K-\ell_2}}{(i+\ell_2+1)\cdots(i+L)} =: \Theta_t^{\ell_2,k_2} V(t,x),$$

where $C_{k_2} := (L+K)^L \left(\frac{1}{1-q_2}\right)^{|k_2-K|}$ and $\Theta_t^{\ell_2,k_2} = C_{k_2} t^{K-k_2-\ell_2} \partial_t^{\ell_2-L} t^{\ell_2}$.

We consider the following equation.

$$V(t,x) = F\left(t, x, \left\{\mathcal{D}_{x,q_1}^{\ell_1,k_1} \Theta_t^{\ell_2,k_2} V(t,x)\right\}_{\Delta_2}\right), \tag{E$_2$}$$

where $F(t,x,\{\xi_{\ell_1 k_1 \ell_2 k_2}\}_{\Delta_2})$ is a majorant function of $f(t,x,\{\xi_{\ell_1 k_1 \ell_2 k_2}\}_{\Delta_2})$ with the following Taylor series

$$F(t,x,\{\xi_{\ell_1 k_1 \ell_2 k_2}\}_{\Delta_2}) = \sum_{p+|\beta|\geq 0} \frac{F_{p\beta}}{(R-x)^{p+|\beta|+1}} t^p \prod_{\Delta_2} (\xi_{\ell_1 k_1 \ell_2 k_2})^{\beta_{\ell_1 k_1 \ell_2 k_2}}. \tag{2.2}$$

Namely, for all (p,β), $f_{p\beta}(x) \ll F_{p\beta}/(R-x)^{p+|\beta|+1}$ hold. In this case, $U(t,x) \ll V(t,x)$ holds.

Here we give the following two lemmas which will be proved later.

Lemma 1 *Let $k \geq 1$ and $\ell \geq 0$. For $R \leq 1/e$, we have*

$$D_{x,q_1}^{\ell} \frac{1}{(R-x)^k} \ll \frac{1}{(eR(1-q_1))^{\ell} R^{\ell k}} \cdot \frac{1}{(R-x)^k}. \tag{2.3}$$

Lemma 2 *Let $V(t,x) = \sum_{i\geq 0} V_i(x) t^i$ be a formal solution of* (E$_2$). *Then for sufficiently small $R > 0$, we can find non negative constants C_{mi} $(0 \leq m \leq (L+K+2)i,\ i \geq 0)$ such that*

$$\begin{aligned} V_0(x) &= \frac{C_{00}}{R-x}, \quad (C_{00} = F_{00}) \\ V_i(x) &\ll \sum_{m=0}^{(L+K+2)i} \frac{C_{mi}}{(R-x)^{m+1}} \quad (i \geq 1). \end{aligned}$$

We continue a proof of Theorem 1 by admitting Lemmas 1 and 2.

We assume $R < 1/e$. By using Lemmas 1 and 2, we have

$$\begin{aligned} &\mathcal{D}_{x,q_1}^{\ell_1,k_1} \Theta_t^{\ell_2,k_2} V(t,x) = \mathcal{D}_{x,q_1}^{\ell_1,k_1} \sum_{i=0}^{\infty} V_i(x) \frac{C_{k_2} t^{i+L-k_2+K-\ell_2}}{(i+\ell_2+1)\cdots(i+L)} \\ &\ll \partial_x^{\ell_1} D_{x,q_1}^{k_1} \sum_{i=0}^{\infty} \sum_{m=0}^{(L+K+2)i} \frac{C_{mi}}{(R-x)^{m+1}} \frac{C_{k_2} t^{i+L-k_2+K-\ell_2}}{(i+\ell_2+1)\cdots(i+L)} \\ &\ll \partial_x^{\ell_1} \sum_{i=0}^{\infty} \sum_{m=0}^{(L+K+2)i} \frac{C_{mi}}{(eR(1-q_1))^{k_1} R^{k_1(m+1)} (R-x)^{m+1}} \frac{C_{k_2} t^{i+L-k_2+K-\ell_2}}{(i+\ell_2+1)\cdots(i+L)} \\ &= \sum_{i=0}^{\infty} \sum_{m=0}^{(L+K+2)i} \frac{(m+1)\cdots(m+\ell_1)}{(i+\ell_2+1)\cdots(i+L)} \frac{C_{mi} C_{k_2}}{(eR(1-q_1))^{k_1} R^{k_1(m+1)}} \frac{t^{i+L-k_2+K-\ell_2}}{(R-x)^{m+1+\ell_1}}. \end{aligned}$$

Since $\ell_1 \leq L+K \leq L+K+2$ and $m \leq (L+K+2)i$, we have

$$m+\ell_1 \le (L+K+2)(i+1) \le (L+K+2)(i+\ell_2+1).$$

In this case, we have

$$\frac{(m+1)\cdots(m+\ell_1)}{(i+\ell_2+1)\cdots(i+L)} \le \frac{(m+\ell_1)^{\ell_1}}{(i+\ell_2+1)^{L-\ell_2}} \le \frac{(L+K+2)^{\ell_1}}{(i+\ell_2+1)^{L-\ell_1-\ell_2}} \le (L+K+2)^{\ell_1},$$

because (E_1) is a "partial differential equation of Kowalevski type". Moreover, we can estimate

$$\frac{C_{mi}C_{k_2}}{(eR(1-q_1))^{k_1}R^{k_1(m+1)}} \le \frac{C_{mi}C_{k_2}}{(eR(1-q_1))^{L+K}R^{(L+K)\{(L+K+2)i+1\}}} =: \tilde{C}_{mi},$$

because $R, eR, 1-q_1$ are less than 1 and $k_1 \le L+K$. Therefore, by putting

$$W(t,x)=\sum_{i=0}^{\infty} W_i(x)t^i, \quad W_i(x)=\sum_{m=0}^{(L+K+2)i} \frac{\tilde{C}_{mi}}{(R-x)^{m+1}},$$

we have

$$\mathcal{D}_{x,q_1}^{\ell_1,k_1}\Theta_t^{\ell_2,k_2}V(t,x) \ll \left(\frac{L+K+2}{R-x}\right)^{\ell_1} t^{L+K-\ell_2-k_2}W(t,x).$$

We consider the following functional equation for $W(t,x)$.

$$\begin{cases} W(t,x)=F\left(t,x,\left\{\left(\dfrac{L+K+2}{R-x}\right)^{\ell_1} t^{L+K-\ell_2-k_2}W(t,x)\right\}_{\Delta_2}\right), \\ W(0,x)=\dfrac{C_{00}}{R-x}. \end{cases}$$

By the implicit function theorem, we obtain a unique holomorphic solution $W(t,x)$ in a neighborhood of the origin. Since $U(t,x) \ll V(t,x) \ll W(t,x)$, we obtain the desired result. □

3. Proof of Theorem 2

Let $U(t,x)=\mathcal{D}_{t,q_2}^{L,K}u(t,x)$ as a new unknown function. Then (E) are written as follows.

$$U(t,x)=f\left(t,x,\left\{\mathcal{D}_{x,q_1}^{\ell_1,k_1}\Lambda_{t,q_2}^{\ell_2,k_2}U(t,x)\right\}_{\Delta_1}\right), \tag{E$_1$}$$

where $\Lambda_{t,q_2}^{\ell_2,k_2}=\partial_t^{\ell_2}D_{t,q_2}^{k_2-K}\partial_t^{-L}$. In a same way as (2.1) in section 2, for $U(t,x)=\sum_{i=0}^{\infty}U_i(x)t^i \ll V(t,x)$, $\Lambda_{t,q_2}^{\ell_2,k_2}U(t,x)$ could be estimated by

$$\begin{aligned} &\Lambda_{t,q_2}^{\ell_2,k_2}U(t,x) \\ &\ll \left(\frac{1}{1-q_2}\right)^{|k_2-K|}\sum_{i=0}^{\infty}V_i(x)\frac{\prod_{r=1}^{\ell_2}(i+L-k_2+K-r+1)}{\prod_{r=1}^{L}(i+r)}t^{i+L-k_2+K-\ell_2}. \end{aligned}$$

We remark that since (E_1) is a "q-difference-differential equations of Kowalevski type ", the power $i+L-k_2+K-\ell_2$ of t is positive and

$$\frac{i+L-k_2+K-r+1}{i+r} \quad\le\quad 1+(L+K-1)=L+K.$$

Moreover, since (E_1) is not a "partial differential equation of Kowalevski type", we can estimate as follows.

• If $\ell_2 < L$,

$$\begin{aligned}\frac{\prod_{r=1}^{\ell_2}(i+L-k_2+K-r+1)}{\prod_{r=1}^{L}(i+r)} &\leq \prod_{r=1}^{\ell_2}\frac{i+L-k_2+K-r+1}{i+r}\prod_{r=\ell_2+1}^{L}\frac{1}{i+r}\\ &\leq (L+K)^L\prod_{r=1}^{L-\ell_2}\frac{1}{i+\ell_2+r}.\end{aligned}$$

In this case,

$$\Lambda_{t,q_2}^{\ell_2,k_2}U(t,x) \ll C_{k_2}\sum_{i=0}^{\infty}V_i(x)\frac{t^{i+L-k_2+K-\ell_2}}{(i+\ell_2+1)\cdots(i+L)} = \Theta_t^{\ell_2<L,k_2}V(t,x), \tag{3.1}$$

where $C_{k_2} := (L+K)^L\left(\frac{1}{1-q_2}\right)^{|k_2-K|}$ and $\Theta_t^{\ell_2<L,k_2} = C_{k_2}t^{K-k_2-\ell_2}\partial_t^{\ell_2-L}t^{\ell_2}$.

• If $\ell_2 = L$,

$$\frac{\prod_{r=1}^{\ell_2}(i+L-k_2+K-r+1)}{\prod_{r=1}^{L}(i+r)} \leq (L+K)^L.$$

In this case,

$$\Lambda_{t,q_2}^{L,k_2}U(t,x) \ll C_{k_2}\sum_{i=0}^{\infty}V_i(x)t^{i-k_2+K} = C_{k_2}t^{K-k_2}V(t,x), \tag{3.2}$$

• If $\ell_2 > L$,

$$\begin{aligned}\frac{\prod_{r=1}^{\ell_2}(i+L-k_2+K-r+1)}{\prod_{r=1}^{L}(i+r)} &\leq (L+K)^L\prod_{r=L+1}^{\ell_2}(i+L-k_2+K-r+1)\\ &\leq (L+K)^L\prod_{r=1}^{\ell_2-L}(i-k_2+K-r+1).\end{aligned}$$

In this case,

$$\begin{aligned}&\Lambda_{t,q_2}^{\ell_2,k_2}U(t,x) \\ &\ll C_{k_2}\sum_{i=0}^{\infty}V_i(x)\prod_{r=1}^{\ell_2-L}(i-k_2+K-r+1)t^{i+L-k_2+K-\ell_2}\\ &= \Theta_t^{\ell_2>L,k_2}V(t,x),\end{aligned} \tag{3.3}$$

where $\Theta_t^{\ell_2>L,k_2} = C_{k_2}\partial_t^{\ell_2-L}t^{K-k_2}$.

Here we define the operator $\Phi_t^{\ell_2,k_2}$ by

$$\Phi_t^{\ell_2,k_2} = \begin{cases}\Theta_t^{\ell_2<L,k_2} & (\ell_2<L),\\ C_{k_2}t^{K-k_2} & (\ell_2=L),\\ \Theta_t^{\ell_2>L,k_2} & (\ell_2>L).\end{cases}$$

We consider the following equation.

$$V(t,x) = F\left(t, x, \left\{\mathcal{D}_{x,q_1}^{\ell_1,k_1}\Phi_t^{\ell_2,k_2}V(t,x)\right\}_{\Delta_1}\right), \tag{E$_3$}$$

where $F(t,x,\{\xi_{\ell_1 k_1 \ell_2 k_2}\}_{\Delta_1})$ is a majorant function of $f(t,x,\{\xi_{\ell_1 k_1 \ell_2 k_2}\}_{\Delta_1})$, which is defined by (2.2). In this case, $U(t,x) \ll V(t,x)$ holds.

We remark that Lemma 2 holds for the equation (E$_3$). We assume $R \leq 1/e$. For the formal solution $V(t,x) = \sum_{i=0}^{\infty} V_i(x)t^i$ of (E$_3$), if $\ell_2 < L$, then we have already known the following majorant relation by using Lemmas 1 and 2.

$$\mathcal{D}_{x,q_1}^{\ell_1,k_1}\Phi_t^{\ell_2,k_2}V(t,x) = \mathcal{D}_{x,q_1}^{\ell_1,k_1}\Theta_t^{\ell_2<L,k_2}V(t,x) \ll \left(\frac{L+K+2}{R-x}\right)^{\ell_1} t^{L+K-\ell_2-k_2}W(t,x),$$

where $W(t,x) = \sum_{i=0}^{\infty} W_i(x)t^i$ with

$$\begin{aligned} W_i(x) &= \sum_{m=0}^{(L+K+2)i} \frac{\tilde{C}_{mi}}{(R-x)^{m+1}}, \\ \tilde{C}_{mi} &:= \frac{C_{mi}C_K}{(eR(1-q_1))^{L+K}R^{(L+K)\{(L+K+2)i+1\}}}. \end{aligned} \tag{3.4}$$

In the case of $\ell_2 > L$,

$$\begin{aligned} &\mathcal{D}_{x,q_1}^{\ell_1,k_1}\Phi_t^{\ell_2,k_2}V(t,x) = \mathcal{D}_{x,q_1}^{\ell_1,k_1}\Theta_t^{\ell_2>L,k_2}V(t,x) \\ &= C_{k_2}\mathcal{D}_{x,q_1}^{\ell_1,k_1}\sum_{i=0}^{\infty} V_i(x)\left(\prod_{r=1}^{\ell_2-L}(i-k_2+K-r+1)\right)t^{i+L-k_2+K-\ell_2} \\ &\ll C_{k_2}\mathcal{D}_{x,q_1}^{\ell_1,k_1}\sum_{i=0}^{\infty}\sum_{m=0}^{(L+K+2)i}\frac{\tilde{C}_{mi}\prod_{r=1}^{\ell_2-L}(i-k_2+K-r+1)}{(R-x)^{m+1}}t^{i+L-k_2+K-\ell_2} \\ &= \sum_{i=0}^{\infty}\sum_{m=0}^{(L+K+2)i}\frac{C_{k_2}\tilde{C}_{mi}A_{mi}}{(R-x)^{m+1+\ell_1}(eR(1-q_1))^{k_1}R^{k_1(m+1)}}t^{i+L-k_2+K-\ell_2}, \end{aligned}$$

where

$$A_{mi} = \prod_{r=1}^{\ell_2-L}(i-k_2+K-r+1)\prod_{r=1}^{\ell_1}(m+r).$$

Here we can estimate A_{mi} as follows.

$$\begin{aligned} A_{mi} &\leq \prod_{r=1}^{\ell_2-L}(i+K)\prod_{r=1}^{\ell_1}((L+K+2)i+\ell_1) \\ &\leq (L+K+2)^{\ell_1+\ell_2-L}(i+1)^{\ell_1+\ell_2-L} = M_0^{\ell_2-L}M_0^{\ell_1}(i+1)^{\ell_1+\ell_2-L}, \end{aligned}$$

where $M_0 = L+K+2$. This implies the following majorant relation by using $W(t,x)$ which is defined as (3.4).

$$\begin{aligned} \mathcal{D}_{x,q_1}^{\ell_1,k_1}\Phi_t^{\ell_2,k_2}V(t,x) &= \mathcal{D}_{x,q_1}^{\ell_1,k_1}\Theta_t^{\ell_2>L,k_2}V(t,x) \\ &\ll M_0^{\ell_2-L}\left(\frac{L+K+2}{R-x}\right)^{\ell_1}t^{L+K-\ell_2-k_2}(t\partial_t+1)^{\ell_1+\ell_2-L}W(t,x). \end{aligned}$$

We define the operator $\Psi_t^{\ell_1,\ell_2,k_2}$ by

$$\Psi_t^{\ell_1,\ell_2,k_2} = \begin{cases} \left(\frac{L+K+2}{R-x}\right)^{\ell_1} t^{L+K-\ell_2-k_2} & (\ell_2 < L), \\ C_{k_2}t^{K-k_2} & (\ell_2 = L), \\ M_0^{\ell_2-L}\left(\frac{L+K+2}{R-x}\right)^{\ell_1} t^{L+K-\ell_2-k_2}(t\partial_t+1)^{\ell_1+\ell_2-L} & (\ell_2 > L), \end{cases}$$

and we consider the following functional equation for $W(t,x)$.

$$\begin{cases} W(t,x) = F\left(t, x, \left\{\Psi_t^{\ell_1,\ell_2 k_2} W(t,x)\right\}_{\Delta_1}\right), \\ W(0,x) = \dfrac{F_{00}}{R-x}. \end{cases} \tag{3.5}$$

This equation is a differential equation in t. We know that the Gevrey order of formal solution of equation (3.5) is estimated by (1.3) (cf. [1] and [2]). Hence Theorem 2 is proved. □

4. Proof of Lemma 1

Let $q_1 = q$ for the simplicity. In order to prove Lemma 1, it is enough to prove the following lemma.

Lemma 3 *Let $k \geq 1$ and $\ell \geq 0$. Then the following majorant relation holds.*

$$D_{q,x}^{\ell} \frac{1}{(R-x)^k} \ll \frac{k^{\ell}}{\{(R(1-q)\}^{\ell}} \cdot \frac{1}{(R-x)^k}. \tag{4.1}$$

By admitting Lemma 3, we can prove Lemma 1 immediately. In fact, if $R < 1$, for any $k \geq 1$, we have $k \leq -1/(eR^k \log R)$, because the maximum of kR^k is $-1/(e \log R)$. Especially if $R \leq 1/e$, we have $k \leq 1/eR^k$.

Proof of Lemma 3. It is trivial for $\ell = 0$.

First, let $\ell = 1$. Since

$$D_{q,x} \frac{1}{R-x} = D_{q,x} \frac{1}{R} \sum_{n \geq 0} \left(\frac{x}{R}\right)^n = \frac{1}{R} \sum_{n \geq 0} \frac{[n]_q}{R} \left(\frac{x}{R}\right)^{n-1}$$

and $[n]_q \leq 1/(1-q)$, we have

$$D_{q,x} \frac{1}{R-x} \ll \frac{1}{R(1-q)} \cdot \frac{1}{R-x}.$$

We assume that when $\ell = 1$, (4.1) holds up to $k-1$. Here we remark that for function $f(x)$ and $g(x)$, we have

$$D_{q,x}\{f(x)g(x)\} = \sigma_q f(x) \cdot D_{q,x} g(x) + D_{q,x} f(x) \cdot g(x), \tag{4.2}$$

where $\sigma_q f(x) = f(qx)$, and if $f(x) \gg 0$, $\sigma_q f(x) \ll f(x)$ holds since $0 < q < 1$. Then we have

$$\begin{aligned} D_{q,x} \frac{1}{(R-x)^k} &= \sigma_q \frac{1}{R-x} \cdot D_{q,x} \frac{1}{(R-x)^{k-1}} + D_{q,x} \frac{1}{R-x} \cdot \frac{1}{(R-x)^{k-1}} \\ &\ll \frac{1}{R-x} \cdot \frac{k-1}{R(1-q)} \frac{1}{(R-x)^{k-1}} + \frac{1}{R(1-q)} \frac{1}{R-x} \cdot \frac{1}{(R-x)^{k-1}} \\ &= \frac{k}{R(1-q)} \cdot \frac{1}{(R-x)^k}. \end{aligned}$$

Next, we assume that for any k, (4.1) holds up to $\ell - 1$. Then we have

$$D_{q,x}^{\ell} \frac{1}{(R-x)^k} \ll D_{q,x} \frac{k^{\ell-1}}{\{R(1-q)\}^{\ell-1}} \cdot \frac{1}{(R-x)^k} \ll \frac{k^{\ell}}{\{R(1-q)\}^{\ell}} \cdot \frac{1}{(R-x)^k}.$$

□

5. Proof of Lemma 2

By substituting $V(t,x) = \sum_{i=0}^{\infty} V_i(x)t^i$ into (E_2), we have the following recurrence formula for $\{V_i(x)\}$.

$$\begin{aligned} V_0(x) &= \frac{C_{00}}{R-x}, \quad (C_{00} = F_{00}), \\ V_i(x) &= \sum_{p+|\beta|\geq 0} \sum{}' \frac{F_{p\beta}}{(R-x)^{p+|\beta|+1}} \prod_{\Delta_1}\prod_{r} \frac{C_{k_2}\mathcal{D}_{x,q_1}^{\ell_1,k_1}\partial_x^{k_1} V_{i_{\ell_1 k_1 \ell_2 k_2 r}}(x)}{(i_{\ell_1 k_1 \ell_2 k_2 r}+1)\cdots(i_{\ell_1 k_1 \ell_2 k_2 r}+j)} \end{aligned}$$

for $i \geq 1$, where $\sum'$ is taken over

$$p + \sum_{\Delta_1}\sum_{r}(i_{\ell_1 k_1 \ell_2 k_2 r} + L - k_2 + K - \ell_2) = i,$$

$\prod_r = \prod_{r=1}^{\beta_{\ell_1 k_1 \ell_2 k_2}}$, $\sum_r = \sum_{r=1}^{\beta_{\ell_1 k_1 \ell_2 k_2}}$ and $C_{k_2} = (L+K)^L \left(\frac{1}{1-q_2}\right)^{|k_2-K|}$.

Lemma 2 can be proved by induction. In order to prove Lemma 2, we calculate the upper bound estimate of the power of $1/(R-x)$ of the majorant function for $V_i(x)$. By Lemma 1,

$$\begin{aligned} \text{power} &\leq p + |\beta| + 1 + \sum_{\Delta}\sum_{r}((L+K+2)i_{\ell_1 k_1 \ell_2 k_2 r} + 1 + \ell_1) \\ &= (L+K+2)\left(p + \sum_{\Delta_1}\sum_{r}(i_{\ell_1 k_1 \ell_2 k_2 r} + L - k_2 + K - \ell_2)\right) + 1 \\ &\qquad + \sum_{\Delta_1}\sum_{r}\{2 + \ell_1 - (L+K+2)(L - k_2 + K - \ell_2)\} - (L+1)p \\ &\qquad\qquad \left(\because \textstyle\sum_{\Delta_1}\sum_{r} 1 = |\beta|\right) \\ &\leq (L+K+2)i + 1 + \sum_{\Delta_1}\sum_{r}\{2 + \ell_1 - (L+K+2)(L - k_2 + K - \ell_2)\} \\ &\leq (L+K+2)i + 1 + \sum_{\Delta_1}\sum_{r}\{2 + L + K - (L+K+2)\} \\ &= (L+K+2)i + 1. \end{aligned}$$

This implies Lemma 2. □

References

[1] Gérard R. and Tahara H., Singular nonlinear partial differential equations, *Vieweg*, 1996.
[2] Shirai A., Maillet type theorem for first order singular nonlinear partial differential equations, *Publ. RIMS, Kyoto Univ.*, **39** (2003), 275–296.